Optimization of Design

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Lecture 2
Lecture outline

- Reading: Ch2 of text

- Today’s lecture
  - Process of transforming the design of a selected system and/or subsystem into an optimum design problem
  - DESIGN OF A CAN
  - INSULATED SPHERICAL TANK DESIGN
The proper definition and formulation of a problem take roughly 50 percent of the total effort needed to solve it.

It is critical to follow well-defined procedures for formulating design optimization problems.

Optimum solution will be only as good as the formulation.

- If we forget to include a critical constraint in the formulation, the optimum solution will most likely violate it.
- If we have too many constraints, or if they are inconsistent, there may be no solution.
THE PROBLEM FORMULATION PROCESS

- Translate a descriptive statement of it into a well-defined mathematical statement.
- Optimization methods are iterative where the solution process is started by selecting a trial design or a set of trial designs. The trial designs are analyzed and evaluated, and a new trial design is generated. This iterative process is continued until an optimum solution is reached.
Five-step formulation procedure

- Step 1: Project/problem description
- Step 2: Data and information collection
- Step 3: Definition of design variables
- Step 4: Optimization criterion
- Step 5: Formulation of constraints
Step 1: Project/Problem Description

- The statement describes the overall objectives of the project and the requirements to be met. This is also called the statement of work.
EXAMPLE 2.1 DESIGN OF A CANTILEVER BEAM

Consider the design of a hollow square-cross-section cantilever beam to support a load of 20 kN at its end. The beam, made of steel, is 2 m long.
The failure conditions for the beam are as follows:

1. The material should not fail under the action of the load.
2. The deflection of the free end should be no more than 1 cm. The width-to-thickness ratio for the beam should be no more than 8. A minimum-mass beam is desired.

The width and thickness of the beam must be within the following limits:

\[
60 \leq \text{width} \leq 300 \text{ mm} \\
10 \leq \text{thickness} \leq 40 \text{ mm}
\]
Step 2: Data and Information Collection

- To develop a mathematical formulation for the problem, we need to gather information on material properties, performance requirements, resource limits, cost of raw materials, and so forth.
- Most problems require the capability to analyze trial designs.
- Therefore, analysis procedures and analysis tools must be identified at this stage.
EXAMPLE 2.2 DATA AND INFORMATION COLLECTION FOR A CANTILEVER BEAM

The information needed for the cantilever beam design problem of Example 2.1 includes expressions for bending and shear stresses, and the expression for the deflection of the free end.

\[
A = w^2 - (w - 2t)^2 = 4t(w - t), \text{ mm}^2
\]

\[
I = \frac{8}{3}wt^3 + \frac{2}{3}w^3t - 2w^2t^2 - \frac{4}{3}t^4, \text{ mm}^4
\]

\[
Q = \frac{3}{4}w^2t - \frac{3}{2}wt^2 + t^3, \text{ mm}^3
\]

\[
M = PL, \text{ N} \cdot \text{mm} \\
V = P, \text{ N}
\]

\[
\sigma = \frac{Mw}{2I}, \text{ N} \cdot \text{mm}^{-2}
\]

\[
\tau = \frac{VQ}{2It}, \text{ N} \cdot \text{mm}^{-2}
\]

\[
q = \frac{PL^3}{3EI}, \text{ mm}
\]
$A$  cross-sectional area, $\text{mm}^2$
$E$  modulus of elasticity, $21 \times 10^4 \text{N} \cdot \text{mm}^{-2}$
$G$  shear modulus, $8 \times 10^4 \text{N} \cdot \text{mm}^{-2}$
$I$  moment of inertia, $\text{mm}^4$
$L$  length of the member, 2000 mm
$M$  bending moment, $\text{N} \cdot \text{mm}$
$P$  load at the free end, 20,000 N
$Q$  moment about the neutral axis of the area above the neutral axis, $\text{mm}^3$
$q$  vertical deflection of the free end, mm
$q_a$  allowable vertical deflection of the free end, 10 mm
$V$  shear force, N
$w$  width (depth) of the section, mm
$t$  wall thickness, mm
$\sigma$  bending stress, $\text{N} \cdot \text{mm}^{-2}$
$\sigma_a$  allowable bending stress, 165 $\text{N} \cdot \text{mm}^{-2}$
$\tau$  shear stress, $\text{N} \cdot \text{mm}^{-2}$
$\tau_a$  allowable shear stress, 90 $\text{N} \cdot \text{mm}^{-2}$
Step 3: Definition of Design Variables

- Identify a set of variables that describe the system, called the design variables.
- The number of independent design variables gives the design degrees of freedom for the problem.

**EXAMPLE 2.3 DESIGN VARIABLES FOR A CANTILEVER BEAM**

- \(w=\) width (depth) of the section, mm
- \(t=\) wall thickness, mm
Step 4: Optimization Criterion

- The criterion must be a scalar function whose numerical value can be obtained once a design is specified.
- Such a criterion is called an objective function for the optimum design problem.
- A criterion needs to be maximized or minimized depending on problem requirements.
- A criterion that is to be minimized is usually called a cost function.
EXAMPLE 2.4 OPTIMIZATION CRITERION FOR A CANTILEVER BEAM

For the design problem in Example 2.1, the objective is to design a minimum-mass cantilever beam. Since the mass is proportional to the cross-sectional area of the beam, the objective function for the problem is taken as the cross-sectional area:

\[ f(w, t) = A = 4t(w - t), \text{ mm}^2 \]
Step 5: Formulation of Constraints

- All restrictions placed on the design are collectively called constraints.
- Many constraint functions have only first-order terms in design variables. These are called linear constraints.
- Linear programming problems have only linear constraints and objective functions.
- More general problems have nonlinear cost and/or constraint functions. These are called nonlinear programming problems.
- A design meeting all requirements is called a feasible design.
Equality and inequality constraints: (a) Feasible region for constraint $x_1 = x_2$ (line A-B); (b) feasible region for constraint $x_1 \leq x_2$ (line A-B and the region above it).
EXAMPLE 2.5 CONSTRAINTS FOR A CANTILEVER BEAM

Bending stress constraint: $\sigma \leq \sigma_a$

$$\frac{PLw}{2I} - \sigma_a \leq 0$$

Shear stress constraint: $\tau \leq \tau_a$

$$\frac{PQ}{2It} - \tau_a \leq 0$$

Deflection constraint: $q \leq q_a$

$$\frac{PL^3}{3EI} - q_a \leq 0$$

Width-thickness restriction: $\frac{w}{t} \leq 8$, $w - 8t \leq 0$

Dimension restrictions

$60 - w \leq 0$, mm; $w - 300 \leq 0$, mm

$3 - t \leq 0$, mm; $t - 15 \leq 0$, mm
Thus the optimization problem is to find $w$ and $t$ to minimize the cost function subject to the eight inequality constraints.
DESIGN OF A CAN

![Diagram of a can with dimensions labeled as H (height) and D (diameter).]
STEP 1: PROJECT/PROBLEM DESCRIPTION

- The purpose of this project is to design a can to hold at least 400 ml of liquid.
- The cans will be produced in the billions, so it is desirable to minimize their manufacturing costs. Since cost can be directly related to the surface area of the sheet metal used, it is reasonable to minimize the amount of sheet metal required.
- Fabrication, handling, aesthetics, and shipping considerations impose the following restrictions on the size of the can:
  - The diameter should be no more than 8 cm and no less than 3.5 cm, whereas the height should be no more than 18 cm and no less than 8 cm.
STEP 2: DATA AND INFORMATION COLLECTION

- Material
- Fabrication, handling, aesthetics, and shipping considerations
STEP 3: DEFINITION OF DESIGN VARIABLES

\[ D = \text{diameter of the can, cm} \]
\[ H = \text{height of the can, cm} \]
STEP 4: OPTIMIZATION CRITERION

\[ S = \pi DH + 2 \left( \frac{\pi}{4} D^2 \right), \text{ cm}^2 \]
STEP 5: FORMULATION OF CONSTRAINTS

$\frac{\pi}{4} D^2 H \geq 400, \text{ cm}^3$  \hspace{1cm} (b)

$3.5 \leq D \leq 8, \text{ cm}$  \hspace{1cm} (c)

$8 \leq H \leq 18, \text{ cm}$

- There are four constraints in Eqs. (c).
- Thus, the problem has two design variables and a total of five inequality constraints.
INSULATED SPHERICAL TANK DESIGN
STEP 1: PROJECT/PROBLEM DESCRIPTION

- The goal of this project is to choose an insulation thickness $t$ to minimize the life-cycle cooling cost for a spherical tank.
- The cooling costs include installing and running the refrigeration equipment, and installing the insulation.
- Assume a 10-year life, a 10 percent annual interest rate, and no salvage value.
- The tank has already been designed having $r$ (m) as its radius.
STEP 2: DATA AND INFORMATION COLLECTION

- To calculate the volume of the insulation material, we require the surface area of the spherical tank, which is given as

\[ A = 4\pi r^2, \text{ m}^2 \]

- To calculate the capacity of the refrigeration equipment and the cost of its operation, we need to calculate the annual heat gain \( G \), which is

\[ G = \frac{(365)(24)(\Delta T)A}{c_1 t}, \text{ Watt-hours} \]
where $\Delta T$ is the average difference between the internal and external temperatures in Kelvin

- $c_1$ is the thermal resistivity per unit thickness in Kelvin-meter per Watt
- $t$ is the insulation thickness in meters.
- $\Delta T$ can be estimated from the historical data for temperatures in the region in which the tank is to be used.

Let $c_2 = \text{the insulation cost per cubic meter ($/m^3$)}$, $c_3 = \text{the cost of the refrigeration equipment per Watt-hour of capacity ($/Wh$)}$, and $c_4 = \text{the annual cost of running the refrigeration equipment per Watt-hour ($/Wh$)}$. 
STEP 3: DEFINITION OF DESIGN VARIABLES

- \( t = \) insulation thickness, m
STEP 4: OPTIMIZATION CRITERION

- The goal is to minimize the life-cycle cooling cost of refrigeration for the spherical tank over 10 years.
- The life-cycle cost has three components: insulation, refrigeration equipment, and operations for 10 years. Once the annual operations cost has been converted to the present cost, the total cost is given as

\[
\text{Cost} = c_2At + c_3G + c_4G
\]

- where \( \text{uspwf} (0.1, 10) = 6.14457 \) is the uniform series present worth factor, calculated using the equation

\[
\text{uspwf}(i,n) = \frac{1}{i} \left[ 1 - (1 - i)^{-n} \right]
\]

- where \( i \) is the rate of return per dollar per period and \( n \) is the number of periods
STEP 5: FORMULATION OF CONSTRAINTS

- Although no constraints are indicated in the problem statement, it is important to require that the insulation thickness be non-negative (i.e., \( t \) greater than 0).
- However, strict inequalities cannot be treated mathematically or numerically in the solution process because they give an open feasible set.
- Therefore, a more realistic constraint is

\[ t \geq t_{\text{min}} \]

- \( t_{\text{min}} \) is the smallest insulation thickness available on the market.
Intermediate formulation

- A summary of the problem formulation for the design optimization of insulation for a spherical tank

**Specified data:** \( r, \Delta T, c_1, c_2, c_3, c_4, t_{\text{min}} \)

**Design variable:** \( t, m \)

**Intermediate variables:**

\[
A = 4\pi r^2
\]

\[
G = \frac{(365)(24)(\Delta T)A}{c_1 t}
\]

\[
\text{Cost} = c_2 At + c_3 G + 6.14457c_4 G
\]

**Constraint:** \( t \geq t_{\text{min}} \)
Final formulation

**EXAMPLE 2.7 FORMULATION OF THE SPHERICAL TANK PROBLEM WITH THE DESIGN VARIABLE ONLY**

Following is a summary of the problem formulation for the design optimization of insulation for a spherical tank in terms of the design variable only:

**Specified data:** $r, \Delta T, c_1, c_2, c_3, c_4, t_{\text{min}}$

**Design variable:** $t, m$

**Cost function:** Minimize the life-cycle cooling cost of refrigeration of the spherical tank,

$$
\text{Cost} = at + \frac{b}{t}, \quad a = 4c_2\pi r^2,
$$

$$
b = \frac{(c_3 + 6.14457c_4)}{c_1}(365)(24)(kT)(4\pi r^2)
$$

**Constraint:**

$$
t \geq t_{\text{min}}
$$
Standard design optimization model

Find an $n$-vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of design variables to

Minimize a cost function:

$$f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$$

subject to the $p$ equality constraints:

$$h_j(\mathbf{x}) = h_j(x_1, x_2, \ldots, x_n) = 0; \quad j = 1 \text{ to } p$$

and the $m$ inequality constraints:

$$g_i(\mathbf{x}) = g_i(x_1, x_2, \ldots, x_n) \leq 0; \quad i = 1 \text{ to } m$$
The number of independent equality constraints must be less than, or at the most equal to, the number of design variables (i.e., \( p \leq n \)),

- When \( p < n \), the optimum solution for the problem is possible.
- When \( p = n \), no optimization of the system is necessary because the roots of the equality constraints are the only candidate points for optimum design.
- When \( p > n \), we have an overdetermined system of equations. No solution for the design problem is possible and the problem formulation needs to be closely reexamined.
- An inequality constraint $g_j(x) \leq 0$ is said to be active at a design point $x^*$ if it is satisfied at equality (i.e., $g_j(x^*)=0$).
- An inequality constraint $g_j(x) \leq 0$ is said to be inactive at a design point $x^*$ if it is strictly satisfied (i.e., $g_j(x^*) < 0$).