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Dziobek configurations of the restricted \((N + 1)\)-body problem with equal masses

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In this article, we count the number of \((N - 1)\)-dimensional central configurations of the restricted \((N + 1)\)-body problem with equal masses. For \(N = 4, 5\), there are 25, 56 central configurations. For \(N = 3\) or \(N \geq 6\), the number of central configurations is \(2^N + N - 1\). © 2012 American Institute of Physics. [http://dx.doi.org/10.1063/1.4732098]

I. INTRODUCTION

Central configurations play an important role in the study of the Newtonian \(n\)-body problem. They determine some special solutions and lead to explicit expression of the solutions. Finding all the central configurations and studying their properties have been known to be challenging in celestial mechanics. Smale listed 18 mathematical problems for the 21 century.\(^{20,21}\) The 6th problem is to prove the finiteness of the number of central configurations in \(\mathbb{R}^2\).

In the 3-body problem, there are 4 central configurations. Three of them are collinear configurations studied by Euler and the other is the equilateral triangle found by Lagrange. These are the only possible central configurations of 3 bodies. For 4 bodies, the finiteness of central configurations in \(\mathbb{R}^2\) was proved by Hampton and Moeckel\(^{15}\) in 2006. For 5 bodies, the finiteness has recently been proved by Albouy and Kaloshin\(^{4}\) in 2012. Smale’s 6th problem is still open for \(n > 5\).

From the point of view of the dimension of configurations, we know that \(n\) bodies form a configuration that spans an affine subspace of dimension at most \(n - 1\). It is proved that, for \(n\) bodies with positive masses, there is only one \((n - 1)\)-dimensional central configuration.\(^{1,22}\) It is when the \(n\) bodies form the regular simplex. However, the \((n - 2)\)-dimensional central configurations are far from understood. They are called the Dziobek configurations.

For \(n = 4\), we know there are finitely many Dziobek configurations from the paper of Hampton and Moeckel.\(^{15}\) There are also some results when some of the masses are equal.\(^{5,11,13,23}\) If we let one of the masses equal to zero, we call the problem the restricted \((3 + 1)\)-body problem. The restricted \((3 + 1)\)-body problem is proved to have 8, 9 or 10 Dziobek configurations.

For \(n = 5\), Kotsireas and Lazard enumerated Dziobek configurations with symmetry in the case of equal masses.\(^{12}\) Considering the restricted \((4 + 1)\)-body problem, we have the result that the number of Dziobek configurations is 25 with the 4 positive equal masses forming a tetrahedron.\(^{2}\) This is counted by knowing the symmetry of these Dziobek configurations.\(^{3}\) The symmetry found in the case of the restricted \((4 + 1)\)-body problem can be generalized to all \(n\).\(^{10}\) In other words, the Dziobek configurations of the \((n + 1)\)-body problem with equal masses forming a regular simplex are symmetrical. In the same paper,\(^{10}\) Leandro also gave upper and lower bounds for the number of Dziobek configurations. In this paper, we will give the exact count of such Dziobek configurations.

Due to the symmetry proved in the paper\(^{10}\) and also the geometry of the configurations, we can derive a polynomial system with 2 equations, 2 variables, and 2 parameters. Counting the positive roots of this system gives the number of Dziobek configurations in one of the symmetric lines. Adding the numbers of Dziobek configurations in all symmetric lines, we get the exact count of the configurations. In Sec. II, we will derive the polynomial system and list main results of this paper.

In Sec. III, we will focus on counting the positive common roots of this system for all parameters of interest. We apply tools such as resultants, Hermite quadratic forms, Sylvester-Habicht sequences
TABLE I. Some polynomials to be manipulated.

<table>
<thead>
<tr>
<th>Polynomials in $\mathbb{Z}[p, q][x]$</th>
<th>Length</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_1 = -p^2 + \cdots + 2p^i q^{26}$</td>
<td>450</td>
<td>26</td>
</tr>
<tr>
<td>$h_{1,x} = p^3 + \cdots + p^3 q^3 x^{17}$</td>
<td>118</td>
<td>17</td>
</tr>
<tr>
<td>$h_{1,x} = -p^2 + \cdots + p^3 q^3 x^{19}$</td>
<td>160</td>
<td>19</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Polynomials in $\mathbb{Z}[p, q][y]$</th>
<th>Length</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_0 = -q^2 + \cdots + 2p^i q^{26}$</td>
<td>450</td>
<td>26</td>
</tr>
<tr>
<td>$h_{1,y} = 2pq^2 + \cdots + p^2 q^3 x^{17}$</td>
<td>118</td>
<td>17</td>
</tr>
<tr>
<td>$h_{1,y} = -q^2 + \cdots + p^2 q^3 x^{19}$</td>
<td>160</td>
<td>19</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Polynomials in $\mathbb{Z}[p, q]$</th>
<th>Length</th>
<th>Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 1073741824p^{24} + \cdots + 14708736q^{21} p^{48}$</td>
<td>1393</td>
<td>69</td>
</tr>
<tr>
<td>$b = 191102976p^{23} + \cdots + 35721p^{13} q^{26}$</td>
<td>1393</td>
<td>69</td>
</tr>
<tr>
<td>$g = -282429536481p^{12} + \cdots + 678113317090881p^{23} q^{45}$</td>
<td>1785</td>
<td>68</td>
</tr>
</tbody>
</table>

and Groebner bases. Many symbolic computations are carried out by the software Mathematica 6.0.0.18. Table I shows the complexity of some polynomials that we manipulate here.

II. PRELIMINARIES AND MAIN RESULTS

The Newtonian $n$-body problem is the study of the dynamics of $n$ point particles with masses $m_i > 0$ and positions $q_i \in \mathbb{R}^d$, moving according to a second order differential equation, called the Newton’s laws of motion:

$$m_j \ddot{q}_j = \sum_{i \neq j} \frac{m_im_j(q_i - q_j)}{r_{ij}^3}, \quad 1 \leq j \leq n,$$

where $r_{ij} = |q_i - q_j|$.

Definition 1: A configuration $q = (q_1, q_2, \ldots, q_n)$ in $\mathbb{R}^d \setminus \triangle$ is called a central configuration if there exists $\lambda > 0$ such that

$$\lambda(q_j - c) + \sum_{i \neq j} \frac{m_i(q_i - q_j)}{r_{ij}^3} = 0, \quad 1 \leq j \leq n,$$

where $c = \frac{1}{M} (m_1 q_1 + \cdots + m_n q_n)$, $M = m_1 + \cdots + m_n$, and $\triangle = \{ q_i = q_j, i \neq j \}$

Let $n = N + 1, d = N - 1$, and $m_{N+1} = 0$. Then the configurations satisfying Eq. (2) are the Dziobek configurations of the $(N + 1)$-body problem. The first $N$ equations involve only the first $N$ bodies and therefore they form a central configuration. Assume that the configuration they form is $(N - 1)$-dimensional. Then these $N$ bodies form a regular simplex configuration, denoted by $S$. Therefore, the number of the Dziobek configurations of this $(N + 1)$-body problem will be the number of the positions $q_{N+1}$ of the $m_{N+1}$ mass where together with $S$ they are a central configuration. The position $q_{N+1}$ satisfies the $(N + 1)$th equation of Eq. (2). With the restriction that the positive masses are equal and that the length of each side of $S$ is 1, we have the following symmetry.10

Theorem 1: The $(N + 1)$th body with zero mass must lie on one of the lines which connect the barycenters of two complementary subsimplices of $S$. Let $p, q$ be the number of vertices of two complementary subsimplices of $S$, and let $x, y$ be the two possible values of the mutual distances from the $(N + 1)$th body to each of the remaining bodies. Then the $(N + 1)$th equation of Eq. (2) becomes

$$(x^2 - y^2)(p + q - \frac{p}{x^3} - \frac{q}{y^3}) = \frac{1}{x^3} - \frac{1}{y^3}.$$
Factoring this equation and canceling the factor \((x - y)\) on both sides, we get the first equation of our polynomial system,

\[f_1 = qx^4 + py^4 + pxy^3 + qx^3 y - (p + q)(x^4 y^3 + x^3 y^4) - x^2 - y^2 - xy = 0.\] (3)

Using mutual distances \(r_{i,j}\) as new coordinates, we have a restriction for \(N + 1\) points in the space of dimension \(N - 1\). The mutual distances must have zero Cayley-Menger determinant. That is,

\[
\begin{vmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & r_{1,2}^2 & \cdots & r_{1,N}^2 & r_{1,N+1}^2 \\
1 & r_{2,1}^2 & 0 & \cdots & r_{2,N}^2 & r_{2,N+1}^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & r_{N,1}^2 & r_{N,2}^2 & \cdots & 0 & r_{N,N+1}^2 \\
1 & r_{N+1,1}^2 & r_{N+1,2}^2 & \cdots & r_{N+1,N}^2 & 0
\end{vmatrix} = 0.
\]

When \(m_1, \ldots, m_N\) forms a regular simplex with side 1, denoting \(r_{N+1,j} = r_{j,N+1} = r_j\) for \(j = 1, \ldots, N\), the Cayley-Menger determinant becomes

\[
\begin{vmatrix}
0 & 1 & 1 & \cdots & 1 & 1 \\
1 & 0 & r_{1,2}^2 & \cdots & r_{1,N}^2 & r_{1,N+1}^2 \\
1 & r_{2,1}^2 & 0 & \cdots & r_{2,N}^2 & r_{2,N+1}^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & r_{N,1}^2 & r_{N,2}^2 & \cdots & 0 & r_{N,N+1}^2 \\
1 & r_{1,2}^2 & \cdots & r_{1,N}^2 & 0
\end{vmatrix} = F(r_1, \cdots, r_N) = 0.
\]

**Lemma 1:** \(F = 0 \iff (N - 1)(1 + \sum_{i=1}^{N} r_i^4) - 2 \sum_{i=1}^{N} r_i^2 - 2 \sum_{i,j=1}^{N} r_i^2 r_j^2 = 0.\)

**Proof:** It is well known that the determinant \(F\) of the Cayley-Menger matrix equals the square of the generalized volume of the \(N\)-dimensional simplex in \(\mathbb{R}^N\) multiplied by a constant. Here the simplex is defined by a regular unit \((N - 1)\)-simplex in \(\mathbb{R}^{N-1}\) and the \((N + 1)\)th position whose distances to the vertices of the regular simplex are \(r_1, \ldots, r_N\). Therefore, the polynomial \(F\) in \(r_1, \ldots, r_N\) is symmetric. Next, by expanding the determinant in the last row, we find that the cofactors consist of at most one column that is not 0’s or 1’s. Therefore, we conclude that the polynomial is of the form

\[f = a + b(r_1^2 + \cdots + r_N^2) + c(r_1^2 r_2^2 + \cdots + r_{N-1}^2 r_N^2) + d(r_1^4 + \cdots + r_N^4).\]

Let \(p_1\) be one of the vertices of the regular unit \((N - 1)\)-simplex, \(p_2\) be the center of this simplex, and \(p_1\) be the intersection point of the regular simplex and the line passing through \(p_1, p_2\). Then we have \(f(p_1) = f(p_2) = f(p_3) = 0\) since the \(N\)-dimensional volume is zero.

At \(p_1\), \((r_1^2, \ldots, r_N^2) = (0, 1, \ldots, 1)\). So \(\sum_{i=1}^{N} r_i^2 = \sum_{i=1}^{N} r_i^4 = N - 1, \sum_{i,j=1}^{N} r_i^2 r_j^2 = (N-1)(N-2)/2\). At \(p_2\), \((r_1^2, \ldots, r_N^2) = (N-1)/2, 1, \ldots, 1)\). So \(\sum_{i=1}^{N} r_i^2 = N-1/2, \sum_{i,j=1}^{N} r_i^2 r_j^2 = (N-1)^3/4N^2\). At \(p_3\), \(r_i^2 = \frac{N}{2(N-1)}, r_i = \frac{N-2}{2(N-1)} \text{ for all } i \geq 2\). So \(\sum_{i=1}^{N} r_i^2 = \frac{N^2-2N+2}{2N-2}, \sum_{i=1}^{N} r_i^4 = \frac{N^4-4N^2+8N-4}{8(N-1)^2}, \sum_{i,j=1}^{N} r_i^2 r_j^2 = \frac{(N-2)(N^2-2N+4)}{8(N-1)}\). Now, we have a linear system

\[
\begin{align*}
a_{11}b + a_{12}c + a_{13}d &= -a, \\
a_{21}b + a_{22}c + a_{23}d &= -a, \\
a_{31}b + a_{32}c + a_{33}d &= -a,
\end{align*}
\]
On the other hand, consider all the points (positive roots when \( p \)) and give the results in Theorem 2. Our basic tools are Groebner bases, Hermite root counting theorem, and Sylvester-Habicht sequences. In Subsections III A–III D, we will introduce these tools first and then demonstrate how to apply them in studying our polynomial system \( f_1 = f_2 = 0 \).

### III. POSITIVE COMMON ROOT COUNTING

In this section, we will count the positive common roots of the polynomial system \( f_1 = f_2 = 0 \) and give the results in Theorem 2. Our basic tools are Groebner bases, Hermite root counting theorem, resultants, and Sylvester-Habicht sequences. In Subsections III A–III D, we will introduce these tools first and then demonstrate how to apply them in studying our polynomial system \( f_1 = f_2 = 0 \).

#### A. Groebner basis

Consider the ideal generated by two polynomials \( f \) and \( g \) in \( \mathbb{C} [x, y] \) and denote it by \( I = \langle f, g \rangle \). On the other hand, consider all the points \((x_0, y_0)\) in \( \mathbb{C}^2 \) that are the common roots of \( f \) and \( g \). We call the set of all these points the variety defined by \( f, g \) and denote it by \( V = V(f, g) \). The ideal \( I \) where \( a_{i,j} \) are functions of \( N \) as given above. Computing the determinant of this system, we get \( \frac{2^N}{N!} \). Therefore, it is not zero for all \( N \geq 2 \). Let \( a = N + 1 \). By Cramer’s rule, we get an unique solution \( (b, c, d) = (-2, -2, N - 1) \).

In our situation, we have \( r_1 = \ldots = r_p = x, r_{p+1} = \ldots = r_N = y \), then we get from Lemma 1 the second equation of our polynomial system,

\[
 f_2 = (p + q - 1) - 2px^2 - 2pqy^2 - 2px^2y^2 + pqx^4 + pqy^4 = 0. 
\]  

The polynomial system \( f_1 = f_2 = 0 \) has 2 variables \( x, y \) and 2 parameters \( p, q \). Note that \( p + q = N \). Therefore, we will count the common positive roots of it for all parameter values with \( p \leq q \in \mathbb{N} \). Each positive root gives a Dziobek configuration position of the zero mass \( m_{N+1} \) on the symmetry line which goes through the centers of a subsimplex with \( p \) vertices and its complementary subsimplex with \( q \) vertices. This position is not in the center of the regular simplex.

**Theorem 2:** The polynomial system \( f_1 = f_2 = 0 \) has 3 positive roots when \( p = 1, q > 1, 4 \) positive roots when \( p = 2, q = 2, 3, \) and 2 positive roots when \( p = 2, q \geq 4 \) and \( p \geq 3, q \geq p \). Moreover, there are no roots with \( x = y \) for those parameters.

Using this theorem, we give the exact count of the number of the Dziobek configurations.

**Theorem 3:** The number of Dziobek configurations of the restricted \((N + 1)\)-body problem with equal masses forming a regular simplex is 25, 56 for \( N = 4, 5, 2^N + N - 1 \) for \( N = 3 \) or \( N \geq 6 \).

**Proof:** We compute the number of the Dziobek configurations for \( N = 3–9 \) in Table II above. Since we factored out \((x - y)\) when forming \( f_1 \) and the polynomial system \( f_1 = f_2 = 0 \) has no solution with \( x = y \), we add 1 to our summations for the case when the \( N + 1 \) zero mass is at the center of the regular simplex. For \( N \geq 6 \), we compute that the number equals to \( 3N + 2(2^{N-1} - 1 - N) + 1 = 2^N + N - 1 \).

\[
 f_2 = (p + q - 1) - 2px^2 - 2pqy^2 - 2px^2y^2 + pqx^4 + pqy^4 = 0. 
\]
Given a total ordering on \( \mathbb{Z}_{\geq 0}^2 \) satisfying \( \alpha + \gamma > \beta + \gamma \) and \( \gamma \geq (0, 0) \) if \( \alpha > \beta \) and \( \gamma \in \mathbb{Z}_{\geq 0}^2 \), we provide the monomials \( x^m y^n \) for all \( m, n \in \mathbb{Z}_{\geq 0} \) and \( \gamma \in \mathbb{Z}_{\geq 0}^2 \). Therefore, by Proposition 1, \( \dim C[\{x, y\}] = \max \{\text{dim } A, \text{dim } V\} \). Thus, \( \dim C[\{x, y\}] = \max \{\text{dim } A, \text{dim } V\} \). If we always have \( \gamma = (0, 0) \), then \( \dim C[\{x, y\}] = \max \{\text{dim } A, \text{dim } V\} \). If we always have \( \gamma = (0, 0) \), then \( \dim C[\{x, y\}] = \max \{\text{dim } A, \text{dim } V\} \). If we always have \( \gamma = (0, 0) \), then \( \dim C[\{x, y\}] = \max \{\text{dim } A, \text{dim } V\} \). If we always have \( \gamma = (0, 0) \), then \( \dim C[\{x, y\}] = \max \{\text{dim } A, \text{dim } V\} \). If we always have \( \gamma = (0, 0) \), then \( \dim C[\{x, y\}] = \max \{\text{dim } A, \text{dim } V\} \). If we always have \( \gamma = (0, 0) \), then \( \dim C[\{x, y\}] = \max \{\text{dim } A, \text{dim } V\} \). If we always have \( \gamma = (0, 0) \), then \( \dim C[\{x, y\}] = \max \{\text{dim } A, \text{dim } V\} \).

In our cases, the polynomials \( f_1, f_2 \) are in \( C[p, q][x, y] \). How can we compute the Groebner bases for all \( p, q \in \mathbb{R} \)? This can be done by the next proposition. We first introduce a monomial ordering called the block monomial ordering. Say we have unknowns \( x_1, x_2, y_1, y_2 \), grouped as \( X, Y \). If we always have \( X > Y \), then we call it a block monomial ordering. Here is the result that we need.

**Definition 2:** A finite subset \( G = \{g_1, \ldots, g_s\} \) of \( I \) is called a Groebner basis of \( I \) if \( \langle \text{LT}(g_1), \ldots, \text{LT}(g_s) \rangle = \langle \text{LT}(I) \rangle \).

**Proposition 1:** Let \( G = \{g_1, \ldots, g_s\} \) be a Groebner basis of \( I \). If \( \exists m, n \in \mathbb{Z}_{\geq 0} \) such that \( x^m = \text{LM}(g_i) \) and \( y^n = \text{LM}(g_j) \) for some \( g_i, g_j \in G \), then

1. the dimension of \( A \) as a vector space over \( C \) is finite,
2. the number of points in \( V \) is at most the dimension of \( A \),
3. a basis of \( A \) is \( \{x^m y^n \} \), and
4. for every \( f \in C[\{x, y\}] \), there exists a unique \( r \in C[\{x, y\}] \) such that no terms of \( r \) are divisible by any of \( \text{LT}(g_1), \ldots, \text{LT}(g_s) \) and \( f = r \in I \). There exists a division algorithm to compute \( r \).

Based on the proposition above, we compute the Groebner bases for all \( p, q > 0 \).

**Proposition 3:** There exists \( GB = \{g_1, \ldots, g_{10}\} \subset C[\{x, y, p, q\}] \) such that, for any \( p_0 > 0, q_0 > 0 \), \( \{g_1(x, y, p_0, q_0), \ldots, g_{10}(x, y, p_0, q_0)\} \) is a Groebner basis of \( J_{p_0, q_0} : = \langle f_1(x, y, p_0, q_0), f_2(x, y, p_0, q_0) \rangle \) in \( C[\{x, y\}] \).

**Proof:** Let \( J = \{f_1, f_2\} \) be the ideal generated by \( f_1, f_2 \) in \( C[\{x, y, p, q\}] \). Let the monomial order be a block order with \( (x, y) > (p, q) \) and use the graded reverse lexicographic order in both \( x, y \) and \( p, q \). We compute the Groebner basis of \( J \) with respect to this order. We get 10 polynomials with leading terms \( pq^4, pq(p + q)x^3 y^4, (p + q)x^4 y^3, q^2 x^4 y^3, 2pq(p + q)x^2 y^3, 2p^2 q(p + q)x^2 y^3, 2q^2(p + q)x^3 y^3, 2pq(p + q)x^3 y^3, 2p^2 q(p + q)x^3 y^3 \). If \( p_0, q_0 > 0 \), none of the leading coefficients vanish. Therefore, by Proposition 2, we know \( \{g_1(x, y, p_0, q_0), \ldots, g_{10}(x, y, p_0, q_0)\} \) is a Groebner basis of \( J_{p_0, q_0} \). And, the set of those 10 polynomials is \( GB \).

**Remark 1:** In the Groebner basis \( \{g_1(x, y, p_0, q_0), \ldots, g_{10}(x, y, p_0, q_0)\} \) of \( J_{p_0, q_0} \), \( x^4 \) and \( y^9 \) are leading monomials. Therefore, by Proposition 1, \( \dim(C[\{x, y\}] / J_{p_0, q_0}) \) is finite. We count the number of monomials that are not divided by \( x^4 \), \( xy^8 \) to get the dimension of \( C[\{x, y\}] / J_{p_0, q_0} \).
are 26 of them. Therefore, \( \dim(\mathbb{C}[x, y]/I_{p_0, q_0}) = 26 \) and there are at most 26 complex zeros of the system \( f_1(x, y, p_0, q_0) = f_2(x, y, p_0, q_0) = 0 \) for \( p_0, q_0 > 0 \).

Here we list a theorem that will be used later and the proof can be found in Ref. 8.

Proposition 4: (Extension theorem) Let \( p_1, p_2 \in \mathbb{C}[x, y] \) and \( f = q_1 p_1 + q_2 p_2 \in \mathbb{C}[y] \) for some \( q_1, q_2 \in \mathbb{C}[x, y] \). Let \( y_0 \) be a root of \( f \). Write \( p_i = c_i^p(y)x^{n_i} + \cdots + c_i^b(y) \) such that \( c_i^p(y) \) are not zero polynomials for \( i = 1, 2 \). If \( c_i^p(y_0) \neq 0 \) for some \( i \), then \( \exists x_0 \in \mathbb{C} \) such that \( p_1(x_0, y_0) = p_2(x_0, y_0) = 0 \).

B. Bifurcation set

By the implicit function theorem, we know when a common root \((x_0, y_0)\) continues in a neighborhood of a given parameter value \((p_0, q_0)\). Such candidate pairs \((x_0, y_0, p_0, q_0)\) are in the variety generated by \( f_1, f_2 \), and the Jacobian polynomial, denoted by \( f_{12} \). We expect to find a polynomial in only \( p \) and \( q \) from a Groebner basis of \( f_1, f_2 \), and \( f_{12} \) in \( \mathbb{C}[x, y, p, q] \) with a monomial order \((x, y) > (p, q)\). This polynomial then defines the equation determining the bifurcation curves in \((p, q)\) plane. However, the computation of such Groebner basis of \( f_1, f_2 \), and \( f_{12} \) is too large to run. We are not able to get the bifurcation curve equation directly from this method. Instead, we use the Hermite’s theorem and two propositions regarding of the intersection multiplicity to find the bifurcation equation.

First, we introduce the Hermite theorem. Let \( p_1, p_2 \in \mathbb{R}[x, y] \), \( I = \langle p_1, p_2 \rangle \) be the ideal generated by \( p_1 \) and \( p_2 \) and \( A = \mathbb{R}[x, y]/I \) be the quotient algebra. Given \( \mathcal{I} \in A \), we define the linear multiplication map on \( A \) by \( L(\mathcal{I})(g) := [fg] \). Let \( \mathcal{I} \in A \). It can be shown that the function defined by \( A \to \mathbb{R} : \mathcal{I} \mapsto \text{Trace}(L([h \mathcal{I}^2])) \) is a quadratic form on \( A \). Given a (vector space) basis of \( A \), say \( \{b_1, \ldots, b_m\} \), then we can form an \( m \times m \) matrix representation of the quadratic form, called the Hermite matrix and denoted by \( H(I, h) \). Each entry \( H(I, h)(i, j) \) is given by \( \text{Trace}(L([h]b_i b_j)) \). From the basic facts about quadratic form, we know the rank and the signature are independent of the choice of the basis. We have the following important real root counting theorem.\(^7\)

The first part will be used to find the bifurcation set. In Sec. III.D, we will use the second part to count the positive zeros of polynomial systems with rational coefficients.

**Theorem 4:** (Hermite)

1. Rank of \( H(I, h) \) equals to the number of common complex root \((x, y)\) of \( p_1 \) and \( p_2 \) such that \( h(x, y) \neq 0 \).
2. Signature of \( H(I, h) \) equals to the number of common real roots of \( p_1 \) and \( p_2 \) such that \( h > 0 \) minus the number of common real roots of \( p_1 \) and \( p_2 \) such that \( h < 0 \).

By part 3 of Proposition 1, we can use a Groebner basis of \( I \) to find a vector space basis of \( A \). \( \{b_1, \ldots, b_m\} \). By part 4 of the same proposition, we can explicitly find the \( m \times m \) real matrix representation of \( L \) with respect to this basis \( \{b_1, \ldots, b_m\} \). In particular, we can explicitly compute the trace. Therefore, we can explicitly compute the \( m \times m \) real matrix \( H(I, h) \). Each entry is the trace of an \( m \times m \) real matrix. We use a Mathematica implementation of this algorithm written by Moecckel and the author of Ref. 16. We can let the computer help us to compute \( H(I, h) \).

Next, recall the definition of intersection multiplicity at the intersection point of two plane curves. Let \( f \) and \( g \) in \( \mathbb{C}[x, y] \), \( I \) be the ideal generated by \( f, g \), and \( V \) be the variety defined by \( f, g \). Assume \( V \) is finite and consists of points \( \{(x_1, y_1), \ldots, (x_k, y_k)\} \). The intersection multiplicity of \( z_i = (x_i, y_i) \) is defined by

\[
\text{mul}(z_i) = \dim(\mathcal{O}_{z_i} : (f, g)_{z_i}),
\]

where \( \mathcal{O}_{z_i} = \left\{ \frac{p}{p_2} \mid p_1, p_2 \in \mathbb{C}[x, y], p_2(z_i) \neq 0 \right\} \) and \( (f, g)_{z_i} = f \mathcal{O}_{z_i} + g \mathcal{O}_{z_i} \).
We need the following two propositions regarding the intersection multiplicity. The proof of the first one can be found in the book.9

Proposition 5: Let f and g in $\mathbb{C} [x, y]$, I be the ideal generated by f, g. Suppose that dim $\mathbb{C} [x, y] / I$ is finite. Then dim $\mathbb{C} [x, y] / I = \sum_{i=1}^{m} \text{mul}(z_{i})$, where $z_{1}, \ldots, z_{m}$ are the complex roots.

Proposition 6: Let f, g $\in \mathbb{C} [x, y]$. If z = (x, y) is a common zero of f, g and the Jacobian polynomial, then mul(z) $\geq 2$.

Proof: Without lost of generality, we assume that z = (0, 0). Denoting $\partial_{x}f|_{z}$ = a and $\partial_{y}f|_{z}$ = b, we can write $f = ax + by + r$ where r contains higher order terms. Since the Jacobean is zero, we have $g = c(ax + by) + r'$. Again, let $O_{z}$ be the local ring and $M_{z} = x O_{z} + y O_{z}$ be the maximal ideal. Then it is easy to see that dim $\frac{(f,g)}{(f,g)I_{z}} \leq 1$. Suppose that mul(z) = dim $\frac{(f,g)}{(f,g)I_{z}} = 1$.

Now, we are ready to find the bifurcation equation of our system $f_{1} = f_{2} = 0$.

Proposition 7: When $p > 1$ and $q > 1$, the bifurcation equation of our system $f_{1} = f_{2} = 0$ is $g(p, q) = -282429536481 p^{12} + \ldots + 678113317090881 p^{23} q^{45}$.

Proof: As in Remark 1, we know that dim $\mathbb{C} [x, y] / J_{p_{0}, q_{0}}$ = 26 for $p_{0}, q_{0} > 0$ and there are at most 26 complex roots for the system $f_{1}(x, y, p_{0}, q_{0}) = f_{2}(x, y, p_{0}, q_{0}) = 0$. On the other hand, it is not hard to claim that the set GB in Proposition 3 is a Groebner basis of $(f_{1}, f_{2}) \subset \mathbb{C} (p, q) [x, y]$. Therefore, we can use GB to compute the Hermite matrix $H(I, 1)$, where $I = (f_{1}, f_{2}) \subset \mathbb{C} (p, q) [x, y]$. Computing the determinant of this matrix, we get

$$det(H) = \frac{549755813888 (p - 1)^{2} (q - 1)^{2} (p + q - 1)^{4} g(p, q)}{p^{38} q^{36} (p + q)^{21}}.$$ 

Since $p > 1$ and $q > 1$, we have det(H) = 0 if and only if $g(p, q) = 0$. If $g(p_{0}, q_{0}) \neq 0$, then det(H) $\neq 0$ and hence the rank of the Hermite matrix $H(J_{p_{0}, q_{0}}, 1)$ is 26. By the first part of the Hermite’s theorem, we know that there are 26 complex roots of the system $f_{1}(x, y, p_{0}, q_{0}) = f_{2}(x, y, p_{0}, q_{0}) = 0$. Also, by Proposition 5, we know that $\sum_{i=1}^{26} \text{mul}(z_{i}) = 26$, where $z_{1}, \ldots, z_{26}$ are the 26 complex roots. Therefore, $\text{mul}(z_{i}) = 1$ for all $i = 1$ to 26. Next, by Proposition 6, we know that $z_{i}$ is not a zero of the Jacobian polynomial $f_{1}$. In conclusion, if $(p_{0}, q_{0}) \in \mathbb{R}^{2}$ such that the number of common roots of $f_{1}$ and $f_{2}$ may change in a neighborhood of the parameter $(p_{0}, q_{0})$, then $g(p_{0}, q_{0}) = 0$.

The plot of $g = 0$ when $1 < p, q < 10$ is given in Figure 1. Only the dots in the figure are the parameters of interest. However, we still study the whole curves in $\mathbb{R}^{2}$ when $p, q > 1$ in Subsection III C to avoid arguing whether the curves pass through the integer points.

Remark 2: Using our algorithm,16 we compute $H(I, 1)$ in less than 1 minute. This is a 26 $\times$ 26 symmetric matrix with entries in $\mathbb{C} (p, q)$. Using the implemented function Det in Mathematica 6.0.0, we need about 4 h and 40 min to compute the determinant of $H$.

In Subsection III C, we can also obtain the bifurcation polynomial g by resultant computations. The results are shown below Theorem 5. There we reduce the polynomial systems of two variables to one polynomial in one variable, $h_{r}$ or $h_{s}$. Then we compute the discriminant of either $h_{r}$ or $h_{s}$ to obtain $g$. However, in this method, we also obtain other irrelevant factors with a power 2, a or b. It is not easy to argue that a and b are irrelevant considering the complexity of these polynomials shown in Table II.
Another advantage of using the first part of the Hermite theorem and intersection multiplicity to find the bifurcation polynomial is that this method can be easily applied to polynomial systems with more than 3 equations and 3 variables. For those systems, resultant method may require more complex computations and produce more irrelevant factors than what we have computed and obtained here for systems with 2 equations and 2 variables.

C. Non-singular points on \( g = 0 \) when \( p, q > 1 \)

In this subsection, we will first recall the resultants and Sylvester-Habicht sequences and then prove some results about behaviors of the common roots of \( f_1 \) and \( f_2 \) nearby a parameter value that are non-singular points on the curve \( g = 0 \).

Definition 3: Let \( f, g \in \mathbb{D}[x] \), where \( \mathbb{D} \) is a domain (in our cases, the domain is a polynomial ring), \( m = \max(\deg(f), \deg(g) + 1) \), \( n = \deg(g) \), and for \( j < n \) the \( j \)th Sylvester matrix of \( f \) and \( g \) is the \( m + n - 2j \) by \( m + n - j \) matrix

\[
Sylv_j(f, g) = \begin{pmatrix}
\vdots & \cdots & a_0 \\
\vdots & \cdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \cdots & \ddots \\
\vdots & \cdots & \ddots & \cdots & \ddots & \cdots \\
b_n & \cdots & a_m & b_0 \\
b_n & \cdots & \ddots & \ddots & \cdots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots \\
\vdots & \cdots & \ddots & \ddots & \ddots & \ddots & \cdots & \ddots \\
b_n & \cdots & \cdots & \cdots & \ddots & \cdots & \cdots & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & b_0
\end{pmatrix}
\]

where \( f(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_0 \) and \( g(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_0 \). Here \( a_m \) may be 0. For \( k = 0, \ldots, m + n - j - 1 \) we let \( Sylv_{j,k}(f, g) \) be the square matrix of dimension
Definition 4: The Sylvester-Habicht sequence associated to \( f \) and \( g \) is \( \text{SyHa}_j(f, g, x) \) for \( j = 0, \ldots, m \), where \( \delta_k = (-1)^{j(k-1)/2} \) and

\[
\text{Sres}_j(f, g, x) = \begin{cases} 
    f, & \text{if } j = m; \\
    g, & \text{if } j = m - 1; \\
    0, & \text{if } n < j < m - 1; \\
    cf_n(g)^m a_n g, & \text{if } j = n; \\
    \sum_{k=0}^j \det(\text{SyH}_j(x, f, g)) x^k, & \text{if } j < n
\end{cases}
\]

where \( cf(g) \) is the coefficient of the term of order \( j \) in \( g(x) \).

Detailed studies of Sylvester-Habicht sequence can be found in Ref. 17. Note that \( \text{SyHa}_j(f, g, x) \) is a polynomial in \( x \) of degree at most \( j \) and \( \text{SyHa}_0(f, g, x) \) is the resultant of \( f \) and \( g \). We denote the resultant by \( \text{Res}(f, g, x) \). Here we just list some important properties that we need.

Theorem 5: (1) \( \text{SyHa}_j(f, g, x) \in \langle f, g \rangle \), the ideal generated by \( f, g \), for all \( j \).
(2) If \( f, g \in \mathbb{C}[x] \), then \( \text{Res}(f, g, x) = 0 \) iff \( f \) and \( g \) have a common root in \( \mathbb{C} \).

We compute the following Sylvester-Habicht sequences and resultants that will be used later.

\[
\text{Res } (f_1, f_2, x) = -(p + q)^2 h_y(x, p, q), \\
\text{Res } (f_1, f_2, y) = -(p + q)^2 h_x(x, p, q), \\
\text{Res}(h_y, \partial_y h_y, y) = cC_1(p, q)a^2(p, q)g(p, q), \\
\text{Res}(h_x, \partial_x h_x, x) = cC_2(p, q)b^2(p, q)g(p, q),
\]

where

\[
c = -4503599627370496, \\
C_1(p, q) = p^{38} q^{55}(p - 1)^8(q - 1)^{14}(1 + 4p + 7p^2)^2(p + q)^{38}, \\
C_2(p, q) = p^{38} q^{55}(p - 1)^8(q - 1)^{14}(1 + 4q + 7q^2)^2(p + q)^{38}.
\]

\[
\text{SyHa}_1(f_1, f_2, x) = p(p + q)y h_{1,y}(x, p, q)x + p(p + q) h_{10,y}(x, p, q). \\
\text{Res}(h_y, h_{1,y}, y) = C_3(p, q)a^2(p, q), \\
\text{Res}(h_y, h_{10,y}, y) = C_4(p, q)a^2(p, q), \\
\text{SyHa}_1(f_1, f_2, y) = q(p + q)x h_{1,x}(x, p, q)y + q(p + q) h_{10,x}(x, p, q). \\
\text{Res}(h_x, h_{1,x}, x) = C_5(p, q)b^2(p, q), \\
\text{Res}(h_x, h_{10,x}, x) = C_6(p, q)b^2(p, q),
\]

where

\[
C_3(p, q) = -p^{62} q^{52}(p - 1)^8(q - 1)^{10}(1 + 4p + 7p^2)^2(p + q)^{30}, \\
C_4(p, q) = -p^{64} q^{54}(p - 1)^8(q - 1)^{12}(1 + 4p + 7p^2)^2(p + q)^{30}, \\
C_5(p, q) = -p^{52} q^{62}(p - 1)^{10}(q - 1)^{8}(1 + 4q + 7q^2)^2(p + q)^{30}, \\
C_6(p, q) = -p^{54} q^{64}(p - 1)^{12}(q - 1)^{8}(1 + 4q + 7q^2)^2(p + q)^{30}.
\]
FIG. 2. Plot of \( h_y(y, q, p_0) = 0 \) near \((y, q) = (y_0, q_0)\). Here \( h_x, h_{1,x}, h_{10,x} \in \mathbb{Z}[p, q][x], h_y, h_{1,y}, h_{10,y} \in \mathbb{Z}[p, q][y], \) and \( g, a, b \in \mathbb{Z}[p, q]. \) More detailed descriptions of them are in the table below. The polynomial \( g \) is our bifurcation equation. Also note that \( C_1, \ldots, C_6 \neq 0 \) for \( p, q > 1. \)

Proposition 8: The polynomial system \( a = b = g = 0 \) has no positive roots.

Proof: We compute two resultants, \( \text{Res}(a, g, q) \) and \( \text{Res}(b, g, q). \) They are polynomials in \( p \) of 3412\(^2\) and 3398\(^2\), respectively. We use the Mathematica command “CountRoots”\(^6\) to separate their positive roots. We find that the union of the open intervals \((0, 5/4), (4/3, 29/30), (3/2, 3), (4, 407/100), (15, 153/10), (223, 1288)\) contains all positive roots of \( \text{Res}(a, g, q) \) and no positive roots of \( \text{Res}(b, g, q). \)

Therefore, \( a, b \) and \( g \) have no common positive roots. \( \square \)

Next, we prove two propositions regarding of non-singular points \((p_0, q_0)\) on \( g = 0. \) The first proposition gives the behavior of the projection of the positive root \((x, y)\) onto \( y\)-coordinate as one of the parameter \( q \) varies near \( q_0 \) and the other one \( p = p_0 \) is fixed. The second proposition extends the existing root behaviors in the \( y\)-coordinate to the common root behaviors of the system \( f_1 = f_2 = 0 \) as one of the parameter \( q \) varies near \( q_0 \) and the other one \( p = p_0 \) is fixed.

Here, we assume \( \partial_x g(p_0, q_0) \neq 0 \) and by Proposition 8 we assume \( a(p_0, q_0) \neq 0. \) There are similar results to these two propositions for the conditions that \((p_0, q_0)\) are non-singular points on \( g = 0 \) satisfying \( \partial_y g(p_0, q_0) \neq 0, a(p_0, q_0) \neq 0 \) or \( \partial_y g(p_0, q_0) \neq 0, b(p_0, q_0) \neq 0 \) or \( \partial_p g(p_0, q_0) \neq 0, b(p_0, q_0) \neq 0. \)

Proposition 9: Let \( p_0 > 1, q_0 > 1 \) such that \( g(p_0, q_0) = 0, \partial_y g(p_0, q_0) \neq 0 \) and \( a(p_0, q_0) \neq 0. \) Suppose there exists \( y_0 > 0 \) such that \( h_y(y_0, p_0, q_0) = \partial_y h_y(y_0, p_0, q_0) = 0. \) Then the polynomial \( h_y(y, q, p_0) \) in \( y \) with parameter \( q \) experiences a saddle node bifurcation at \( q = q_0 \) and \( y = y_0. \) Figure 2 illustrates the geometry of zeros of \( h_y(y, q, p_0) \) near \((y_0, q_0).\)
Proof: Denote \( f(y, q) = h_y(y, q, p_0) \) and \( r(q) = \text{Res}(f, q) \). We have \( f(y_0, q_0) = \delta_y f(y_0, q_0) = 0 \) and \( r(q_0) = cC_1(p_0, q_0)\alpha^2(p_0, q_0)g(p_0, q_0) = 0 \).

\[
r'(q_0) = c\delta_y C_1(p_0, q_0)\alpha^2(p_0, q_0)g(p_0, q_0) + 2cC_1(p_0, q_0)\alpha(p_0, q_0)\delta_y \alpha(p_0, q_0)g(p_0, q_0) + cC_1(p_0, q_0)\alpha^2(p_0, q_0)\delta_y g(p_0, q_0) = cC_1(p_0, q_0)\alpha^2(p_0, q_0)\delta_y g(p_0, q_0) \neq 0.
\]

Therefore, \( f \) and \( \partial_y f \) intersect at \((y_0, q_0)\) transversely. Therefore,

\[
\begin{vmatrix}
\partial_y f & \partial_y g \\
\partial_{yy} f & \partial_{yy} g
\end{vmatrix} = \begin{vmatrix} 0 & \partial_y g \\ \partial_{yy} f & \partial_{yy} g \end{vmatrix} \neq 0 \Rightarrow \partial_{yy} f \neq 0, \partial_y f \neq 0.
\]

\[\square\]

Proposition 10: Let \( p_0 > 1, q_0 > 1 \) such that \( g(p_0, q_0) = 0 \) and \( \partial_y g(p_0, q_0) \neq 0 \) and \( a(p_0, q_0) \neq 0 \). Suppose there exist \( x_0 > 0, y_0 > 0 \) such that \( f_1(x_0, y_0, p_0, q_0) = f_2(x_0, y_0, p_0, q_0) = 0 \) and the Jacobian polynomial \( f_2(x_0, y_0, p_0, q_0) = 0 \). Let \( p = p_0 \) be fixed. Then the positive common root \((x_0, y_0)\) splits continuously into 2 positive roots in a half neighborhood of \( q_0 \).

Proof. At \((p, q) = (p_0, q_0)\), since \( f_1 \) and \( f_2 \) intersect at \((x_0, y_0)\) not transversely, we have \( h_y(y_0, p_0, q_0) = \delta_y h_y(y_0, q_0) = 0 \). By Proposition 9, the positive root \( y_0 \) splits continuously into \( y_1(q) \) and \( y_2(q) \). They are defined in a half neighborhood of \( q_0 \). Next, we claim that each \( y_1(q) \) and \( y_2(q) \) can be uniquely extended to a common positive root of \( f_1 \) and \( f_2 \) such that the \( x \) coordinates are also continuous in \( q \). Note first that since \( p_0q_0 \neq 0 \), the leading coefficient of \( f_2 \) as a polynomials in \( \mathbb{C}[y][x] \) is non-zero. Therefore, by Proposition 4, each \( y_1(q) \) and \( y_2(q) \) can be extended to a common complex root.

Next, we look at \( SyHa_1(f_1, f_2, x) = p(p + q)y_{h_{1, y}}(y, p, q)x + p(p + q)h_{10, y}(y, p, q) \). We have \( \text{Res}(h_{1, y}, h_{10, y}, y) = C_3(p, q)\alpha^2(p, q) \). Therefore, \( \text{Res}(h_{1, y}, h_{10, y}, y)(p_0, q_0) \neq 0 \). These mean that \( h_{1, y}(y_0, p_0, q_0), h_{10, y}(y_0, p_0, q) \neq 0 \) at and hence in a neighborhood of \( q_0 \).

Finally, defining \( x_1(q) \) by solving \( SyHa_1(f_1, f_2, x)(x, y_1(q)) \) and \( x_2(q) \) by solving \( SyHa_1(f_1, f_2, x)(x, y_2(q)) \), we get two sets of positive common roots \((x_1(q), y_1(q)) \) and \((x_2(q), y_2(q)) \) which are continuous in a half neighborhood of \( q_0 \).

\[\square\]

D. Proof of Theorem 2

Now we are ready to prove our main result of Theorem 2.

Proof. Recall that the polynomial system is

\[
\begin{align*}
f_1 &= qx^4 + py^4 + px^2y^3 + qx^3y - (p + q)(x^4y^3 + x^3y^4) - x^2 - y^2 - xy = 0, \\
f_2 &= (p + q - 1) - 2px^2 - 2qy^2 - 2pqx^2y^2 + pqx^4 + pqy^4 = 0.
\end{align*}
\]

Our goal is to count the number of positive common roots of this system for all \( p, q \in \mathbb{N} \) and \( q \geq p > 1 \) or \( q > p = 1 \), and show among roots \( x \neq y \) for those \( p, q \).

1. The region of \( p \geq 9, q \geq p \)

First, we compute the resultant \( \text{Res}(g, \partial_y g, q) \). This is a polynomial in \( p \) of 3330th. Using the Mathematica command “CountRoots,”\(^{46}\) we find that it has no real root for \( p \geq 9 \). We find that \( g(9, q) = 0 \) only has one real root which is greater than or equal to 9 and also that \( g(q, q) = 0 \) has no real root for \( q \geq 9 \). In conclusion, we know that in the region of \( q \geq p, p \geq 9 \), there are only two open connected sets separated by the curve \( g = 0 \). Since the real zero of \( g(9, q) = 0 \) for \( q \geq 9 \) is approximately 32.7, we can pick two points \((9, 30)\) and \((9, 40)\) as representative points from the two open connected regions. By the implicit function theorem, if we count the number of real roots at these two parameters, we know the number of real roots at these two regions. Also it is easy to
see that when \( p, q > 1 \), the system \( f_1 = f_2 = 0 \) has no real root with zero in one of the coordinates. Therefore, the number of positive roots stays the same in the open connected regions.

2. \( g \neq 0 \) in the region of \( p \geq 9, q \geq p \)

Now, we demonstrate the positive root counting by the Hermite’s theorem at \( (p, q) = (9, 30) \). Let \( I = \langle f_1, f_2 \rangle \) be the ideal generated by \( f_1, f_2 \) in \( \mathbb{C}[x, y] \) when \( (p, q) = (9, 30) \). Using the graded reverse lexicographic order in \( x, y \). We compute the Groebner basis with respect to this order. We get 10 polynomials with leading terms \( 270x^4, 10530x^3y^4, \ldots, 10530xy^8 \) for \( I \). By Proposition 1, we count the number of monomials that are not divided by \( x^4, \ldots, xy^8 \) to get the dimension of \( \mathbb{C}[x, y]/I \). There are 26 of them. Therefore, the dimension of is 26. Next, we compute four Hermite matrices \( H(I, 1), H(I, x), H(I, y), H(I, xy) \). Their signatures are 4, 2, 2, 0. Let \( a_1, a_2, a_3, a_4 \) denote the number of common real roots with signs \((+, +), (+, -), (-, +), (-, -)\), respectively. We get

\[
\begin{align*}
4 & = a_1 + a_2 + a_3 + a_4, \\
2 & = a_1 + a_2 - a_3 - a_4, \\
2 & = a_1 - a_2 + a_3 - a_4, \\
0 & = a_1 - a_2 - a_3 + a_4.
\end{align*}
\]

Therefore, we get \((a_1, a_2, a_3, a_4) = (2, 1, 1, 0)\). So, there are 2 positive common roots of \( f_1 \) and \( f_2 \) when \( (p, q) = (9, 30) \). Similarly, we get that \( f_1 \) and \( f_2 \) have also 2 positive common roots when \( (p, q) = (9, 40) \). Therefore, generically, there are 2 positive common roots of \( f_1 \) and \( f_2 \) when \( q \geq p, p \geq 9 \).

3. \( g = 0 \) in the region of \( p \geq 9, q \geq p \)

Next, we will argue that the system still has 2 positive for parameters in this region on the curve \( g = 0 \). Since \( \text{Res}(g, \partial_q g, q) \neq 0 \), we have \( \partial_q g \neq 0 \). Also, since \( g = 0 \), we assume \( a \neq 0 \) by Proposition 8. Therefore, we can apply Propositions 9 and 10. Suppose there are 3 positive common roots. Fixing \( p = p_0 \), then there is at least one intersection point where the Jacobian polynomial is zero. If not, then \( f_1 = 0 \) and \( f_2 = 0 \) intersect transversely at all 3 points which can be continued in a neighborhood of \( q_0 \). This contradicts the fact that generically there are only 2 positive common roots. However, if \( f_1 = 0 \) and \( f_2 = 0 \) intersect not transversely at one point, by Lemmas 9 and 10, such a positive root can split continuously into 2 positive common roots, which contribute to get totally 4 positive roots in a half neighborhood of \( q_0 \) and is again a contraction. Cases of more than 3 positive roots are clearly not possible. Therefore, \( f_1 = f_2 = 0 \) has at most 2 positive roots on the bifurcation curve \( g = 0 \) in the region \( q \geq p, p \geq 9 \).

We next claim that the system has at least 2 positive roots and hence conclude that there are exactly 2 positive roots on the curve \( g = 0 \) in the region \( q \geq p, p \geq 9 \). Let \((p_0, q_0)\) be on the curve \( g = 0 \) in the region \( q \geq p, p \geq 9 \). Let us look at the resultant \( \text{Res}(f_1, f_2, x) = -(p + q)h_y(y, p, q) \) and the polynomial \( h_y(y, p, q) \). If there is no positive real root at \((p_0, q_0)\), fixing \( p = p_0 \), we know from continuity that the graph of \( h_y(y, p_0, q) \) in \( y - y \) plane must have \( q = q_0 \) as its vertical asymptote. But, this cannot happen since the leading coefficient of \( h_y(y, p_0, q) \) as a polynomial in \( y \) is \( 2p_0^3q^4(p_0 + q)^3 \). It does not approach zero when \( q \) approaches \( q_0 \). Therefore, \( y \) cannot go to infinity as \( q \) approaches \( q_0 \). Finally, suppose there is only one positive real root at \((p_0, q_0)\), excluding the possibility that \( y \) goes to infinity as \( q \) approaches \( q_0 \), we must have that the curve of \( h_y(y, p_0, q) = 0 \) intersect at one point \((y_0, q_0)\) like a cross shape where \( y_0 \) is the \( y \) coordinate of the positive real root. But, this contradicts to the fact that \( h_y(y, q, p_0) \) in \( y \) with parameter \( q \) experiences a saddle node bifurcation at \( q = q_0 \) and \( y = y_0 \) in Proposition 9.

4. \( p = 2, \ldots, 8, q \geq p \)

Finally, what left are the cases when \( p = 1, \ldots, 8, q \geq p \). We claim in the Proposition 7 that \( g = 0 \) is the bifurcation curve for \( p, q > 1 \). For \( p = 1 \), we need to discuss this case separately. For \( p = 2 \), we find that \( g(2, q) = 0 \) has 2 positive roots greater than 2. They are approximately 3.31 and 6.69. So we count the positive roots at \((p, q) = (2, 3), (2, 5)\), and \((2, 7)\) to find that there are 4, 2, 2...
positive roots, respectively. Therefore, the system \( f_1 = f_2 = 0 \) has 4 positive roots at \((2, 2), (2, 3)\) and has 2 positive roots at \((2, q)\) for all \(q \geq 4\). For \(p = 3, \ldots, 8\), we find that \(g(p, q) = 0\) has all one root greater than or equal to \(q\). We find two sample points in each case and that the number of positive roots are all 2 at those points. Therefore, we conclude that the system has 2 positive roots for all \(p = 3, \ldots, 8\) and \(q \geq p\).

\section{5. \(p = 1, q > p\)}

Now, for \(p = 1\), we compute a Groebner basis of \(f_1, f_2\), and the Jacobian polynomial \(f_{12}\) in \(\mathbb{C}[x, y, q]\) with monomial order \(x, y > q\). There is a polynomial in this basis of the form \(h(q)x\), where \(h\) is a polynomial in \(q\) of degree 18. Since we ask for non-zero solution, we get \(h(q_0) = 0\) if \(f_1\) and \(f_2\) intersect not transversely at some point at the parameter \((p, q) = (1, q_0)\). Therefore, this \(h\) is our bifurcation polynomial in \(q\) for \(p = 1\). We find that \(h(q) = 0\) has no root that is greater than 1. Using the Hermite’s theorem, we count at one simple parameter \((p, q) = (1, 2)\) the positive roots of the system \(f_1 = f_2 = 0\) and find there are 3 positive roots. Therefore, there are 3 positive roots for all \(p = 1, q > 1\).

\section{6. \text{No roots with } x = y}

Finally, we prove that the system \(f_1 = f_2 = 0\) has no common roots with \(x = y\) for all \(p, q \in \mathbb{N}\). Let \(x = y\) in the polynomials \(f_1\) and \(f_2\). We get the two polynomials in \(y, -y^2(3 - 2p^2 - 2qy^2 + 2py^3 + 2qy^4) - 1 \leq p + q - 2py^2 - 2qy^2\). Computing their resultant with respect to \(y\), we get a polynomial in \(k(p, q) = -4(p + q)^2(a_0(p) + a_1(p)q + a_2(p)q^2 + a_3(p)q^3 + a_4(p)q^4 + 7q^5), a_0 = 1 - 5p + 10p^2 + 118p^3 - 59p^4 + 7p^5, a_1 = -5 + 20p + 354p^2 - 236p^3 + 35p^4, a_2 = 10 + 354p - 354p^2 + 70p^3, a_3 = 118 - 236p + 70p^2, a_4 = -59 + 35p\).

We find that \(a_0, \ldots, a_4\) have no real roots for \(p \geq 6\) and they are all positive at \(p = 6\). Therefore, \(a_0, \ldots, a_4 > 0\) for \(p \geq 6\). So, \(k(p, q) < 0\) for \(p \geq 6, q > 0\). Also for \(p = 1, 2, 3, 4, 5\, k(p, q) = 0\) has no positive integer zero. In conclusion, \(k(p, q) \neq 0\) for all \(p, q \in \mathbb{N}\). This means that the system \(f_1 = f_2 = 0\) has no common roots with \(x = y\) for all \(p, q \in \mathbb{N}\). □

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16 See http://www.math.umn.edu/~tsaix066/Hermite.nb for implementation of the algorithm of Hermite matrix in Mathematica 6.0.0.