Estimating the number of tetrahedra determined by volume, circumradius and four face areas using Groebner basis

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ABSTRACT

Given any set of six positive parameters, the number of tetrahedra, all having these values as their volume, circumradius and four face areas, is studied. We identify all parameters that determine infinitely many tetrahedra. On the other hand, we classify parameters that determine finitely many tetrahedra and find only four different upper bounds, zero, six, eight, and nine, on the numbers of tetrahedra. In each case, the upper bound is sharp in the complex domain.

In this paper, the upper bounds are obtained through checking the dimensions of various quotient algebras of ideals by counting monomials. This is done by computing Groebner bases with block orders. Partitioning the parameter space into several cases, we find either the dimension or an upper bound of it for the quotient algebra in each case. From that, various upper bounds on the number of tetrahedra are obtained. To show the upper bounds are sharp, we pick rational parameters and study the number of tetrahedra through Hermite’s root counting method.

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1. Introduction

In 1999, M. Mazur defined a tetrahedron to be rigid if there are no other tetrahedra with the same volume, radius of circumscribed sphere, and areas of faces (Mazur, 1999). He proposed a question asking whether every tetrahedron is rigid. Specifically, for given six positive constants $V, R, A_1, A_2, A_3, A_4$, when there is a tetrahedron having these values as its volume, circumradius and four face areas, respectively, is such tetrahedron unique?

This question was answered negatively by Lisoněk and Israel in 2000 (Lisoněk and Israel, 2000). They found that, when $V = \sqrt{\frac{1}{3}}, R = \sqrt{\frac{7}{12}}, A_1 = A_2 = \sqrt{\frac{7}{16}}, A_3 = A_4 = \frac{1}{2}$, there are two different tetrahedra having these values as their volume, circumradius and four face areas, respectively. They are shown in Fig. 1, where the tetrahedron on the left side has squares of edge lengths 1, 2, 1, 1, 2, 2 and the other tetrahedron has squares of edge lengths $\alpha_1, \beta_1, \beta_2, \beta_1, \gamma_1$ with $\alpha_1 \approx 0.59$, $\beta_1 \approx 1.71$, $\beta_1 \approx 2.09$, $\gamma_1 \approx 0.69$ and vertices $v_1 = (0, 1.5, 0), v_2 \approx (0.77, 1.5, 0), v_3 \approx (0.14, 2.8, 0), v_4 \approx (0.63, 2.6, 0.65)$.

In the same paper by Lisoněk and Israel, they posed more questions. They ask whether for any positive constants $V, R, A_1, A_2, A_3, A_4$, there are finitely many tetrahedra, all having these values as their respective metric invariants. And, if there are finitely many tetrahedra, what are the upper bounds?

In 2005, Yang and Zeng presented a negative solution in Yang and Zeng (2005) and Mucherino et al. (2013). When $V = 441, R = \frac{41\sqrt{3}}{6}, A_1 = 84\sqrt{3}, A_2 = A_3 = A_4 = 63\sqrt{3}$, they found a family of tetrahedra $T_{(x,y)}$, where $(x, y)$ varies over a component of a cubic curve such that all tetrahedra $T_{(x,y)}$ share the same metric invariants. The example they proposed is in the case of $A_2 = A_3 = A_4$. They made a conjecture that, when $A_1, A_2, A_3, A_4$ are pairwise distinct, such metric invariants indeed determine finitely many tetrahedra. They also conjectured an upper bound nine in these cases.

Yang and Zeng, in 2013, proved their conjectures by claiming that, given six positive numbers $V, R, A_1, A_2, A_3, A_4$, there are at most eight tetrahedra with volume $V$, circumradius $R$ and four face areas $A_1, A_2, A_3, A_4$, except in the case that three of the values $A_1, A_2, A_3, A_4$ are equal (Yang and Zeng, 2013). The upper bound eight is obtained in the real domain. The cases of three equal face areas are not discussed there.

In McConnell (2012), McConnell introduced the concept of pseudo-faces and used a different system from that in Lisoněk and Israel (2000) and Yang and Zeng (2005) to study the same problem. There, the example of Yang and Zeng in 2005 is generalized. Also, finiteness results are proved in the cases of $A_1 = A_2 = A_3 = A_4$ and $A_1 = A_2 \neq A_3 = A_4$. However, the cases with only two equal face areas are not discussed. Also, none of the upper bounds are given in the paper McConnell (2012).

In this paper, we use a parametric polynomial system with three pseudo-faces introduced by McConnell as variables and $V, R, A_1, A_2, A_3, A_4$ as parameters to study the numbers of tetrahedra. The method used here is different from those in Yang and Zeng (2013) and McConnell (2012). We mainly apply Groebner basis in studying our parametric polynomial system. Both the finiteness
or the upper bounds are obtained through checking the dimensions of various quotient algebras of certain ideals. For some rational parameters, we study the numbers of tetrahedra through Hermite’s root counting method.

Our results are more complete than those in Yang and Zeng (2013) and McConnell (2012). We obtain necessary and sufficient conditions on the parameters that determine infinitely many tetrahedra. We provide upper bounds of the common zeros in the complex domain (and hence the upper bounds of the tetrahedra) when the parameters determine finitely many tetrahedra. For those parameters, partitioning the parameter space into several cases, we found four different upper bounds, zero, six, eight, and nine, on the numbers of tetrahedra. In each case, the upper bound is sharp in the complex domain.

In the next section, we will reduce the problem to root counting problems and state our main theorem. In Section 3, we will present the Groebner basis tools and some useful theorems regarding the zeros of a polynomial system. In Section 4, we prove our main theorem. In Section 5, we conclude this paper and make some remarks on the future studies. We choose Mathematica 10 as our Computer Algebra System to perform many symbolic computations. In Tsai (2015), we provide a link for a Mathematica notebook containing results of our Groebner basis computations.

2. Main results

2.1. Using six mutual distances as the variables

Given a tetrahedron $T$ with vertices $P_1$, $P_2$, $P_3$, $P_4$, we denote the six edge lengths by $d_{i,j}$, for $i < j \in \{1, 2, 3, 4\}$. Given six positive constants $(d_{1,2}, d_{1,3}, d_{1,4}, d_{2,3}, d_{2,4}, d_{3,4})$, if there is a tetrahedron with these values as their mutual distances, such tetrahedron is unique up to rigid motions in $\mathbb{R}^3$.

We can express the volume $V$, circumradius $R$, and four face areas $A_1$, $A_2$, $A_3$, $A_4$ in terms of $d_{i,j}$’s. The Cayley-Menger determinant associated with $P_1$, $P_2, P_3, P_4$, is the determinant of the following $5 \times 5$ matrix.

$$
M = \begin{pmatrix}
0 & d_{1,2}^2 & d_{1,3}^2 & d_{1,4}^2 & 1 \\
d_{1,2}^2 & 0 & d_{2,3}^2 & d_{2,4}^2 & 1 \\
d_{1,3}^2 & d_{2,3}^2 & 0 & d_{3,4}^2 & 1 \\
d_{1,4}^2 & d_{2,4}^2 & d_{3,4}^2 & 0 & 1 \\
1 & 1 & 1 & 1 & 0
\end{pmatrix}
$$

Let $M_0$ denote the determinant of $M$, and $M_i$ denote the principal minor determinant of $M$ obtained by deleting the $i$-th row and $i$-th column of $M$ for $i = 1, \ldots, 5$. Then we have the following equations.

$$
288V^2 - M_0 = 0, \\
2M_0R^2 + M_5 = 0, \\
16A_1^2 + M_1 = 0, \\
16A_2^2 + M_2 = 0, \\
16A_3^2 + M_3 = 0, \\
16A_4^2 + M_4 = 0.
$$

The six equations above give necessary conditions for the existence of a tetrahedron with the corresponding metric invariants. They are also sufficient conditions as well (Blumenthal, 1970). That is, given six positive constants $(V, R, A_1, A_2, A_3, A_4)$, if there exist six positive numbers $(d_{1,2}, d_{1,3}, d_{1,4}, d_{2,3}, d_{2,4}, d_{3,4})$ such that the twelve positive numbers satisfy the six equations above, then there exists a tetrahedron with those corresponding metric invariants.
Here, we are first given six positive constants \((V, R, A_1, A_2, A_3, A_4)\), and want to find the number of tetrahedra with those metric invariances. The six equations above provide us a parametric polynomial system. The problem now is reduced to counting the number of positive common zeros \((d_{1,2}, d_{1,3}, d_{1,4}, d_{2,3}, d_{2,4}, d_{3,4})\) of this system for all positive parameters \(V, R, A_1, A_2, A_3, A_4\). In Lisoněk and Israel (2000) and Yang and Zeng (2005), such system is used to tackle the problem. In this paper, we use a different system introduced in McConnell (2012).

### 2.2. Using three pseudo-faces as the variables

In the previous subsection, the system consists of six equations and six variables of squares of edge lengths. Here, we present another system that has three equations and three variables in three pseudo-faces introduced in McConnell (2012). This new system may provide less intuitive point of view in studying our problem at the first glance, but the three pseudo-faces indeed have good relations with six edge lengths. Therefore, it provides an easier and better system to study our problems.

Let’s first summarize some notations. Given a tetrahedron \(T\) with vertices \(P_1, P_2, P_3, P_4\), we denote the six edge lengths by \(d_{i,j}\) and six dihedral angles by \(\alpha_{i,j}\), for \(i < j \in \{1, 2, 3, 4\}\). We denote the volume by \(V\), circumradius by \(R\), and four face areas by \(A_1, A_2, A_3, A_4\), as in the previous subsection. Define three pseudo-faces \(H, J, K\) by the following equations.

\[
\begin{align*}
A_1^2 + A_2^2 - 2A_1A_3\cos \alpha_{2,4} &= H^2 = A_1^2 + A_2^2 - 2A_2A_4\cos \alpha_{1,3} \\
A_1^2 + A_4^2 - 2A_1A_4\cos \alpha_{2,3} &= J^2 = A_1^2 + A_3^2 - 2A_2A_3\cos \alpha_{1,4} \\
A_2^2 + A_4^2 - 2A_3A_4\cos \alpha_{1,2} &= K^2 = A_1^2 + A_2^2 - 2A_1A_2\cos \alpha_{3,4}
\end{align*}
\]

Geometrically, \(H\) is the area of the quadrilateral when projecting \(T\) into a plane parallel to opposite edges \(d_{1,3}, d_{2,4}\). When projecting \(T\) into such a plane, the quadrilateral has diagonals \(d_{1,3}, d_{2,4}\). Therefore, by Bretschneider’s formula, we have the following expressions for \(H, J, K\) in terms of \(d_{i,j}\)’s.

\[
\begin{align*}
16H^2 &= 4(d_{1,3}^2 + d_{2,4}^2) - (d_{1,2}^2 + d_{3,4}^2 - d_{1,4}^2 - d_{2,3}^2)^2. \\
16J^2 &= 4(d_{1,4}^2 + d_{2,3}^2) - (d_{1,2}^2 + d_{3,4}^2 - d_{1,3}^2 - d_{2,4}^2)^2. \\
16K^2 &= 4(d_{1,2}^2 + d_{3,4}^2) - (d_{1,3}^2 + d_{2,4}^2 - d_{1,4}^2 - d_{2,3}^2)^2.
\end{align*}
\]

On the other hand, we can also express six edge lengths in terms of \(H, J, K\) by the following identities.

\[
\begin{align*}
9V^2d_{1,2}^2 &= [K, A_3, A_4], \\
9V^2d_{3,4}^2 &= [K, A_1, A_2], \\
9V^2d_{1,3}^2 &= [H, A_2, A_4], \\
9V^2d_{2,4}^2 &= [H, A_1, A_3], \\
9V^2d_{2,3}^2 &= [J, A_1, A_4], \\
9V^2d_{2,4}^2 &= [J, A_2, A_3],
\end{align*}
\]

where \([a, b, c]\) denotes the product \((a + b + c)(b + c - a)(a + c - b)(a + b - c)\).

By the two sets of expressions above, we know, given positive \(V, R, A_1, A_2, A_3, A_4\), the number of triple \((H, J, K)\) determined by them is finite if and only if the number of sextuple \((d_{i,j})\) determined by them is finite. In the cases of finiteness, the numbers determined are the same. In contrast to the six equations involving \(d_{i,j}\)’s, the three equations involving \((H, J, K)\) are the following (McConnell, 2012).
Table 1

Upper bounds and examples (N: the equation does not hold; Y: the equation holds).

<table>
<thead>
<tr>
<th>Distinct face areas</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$u$</th>
<th>Example(s) that realize(s) the upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1 = a_2 = a_3 = a_4$</td>
<td>N/Y</td>
<td>N/Y</td>
<td>6</td>
<td>$(\frac{\sqrt{7}}{3}, \frac{\sqrt{7}}{3}, \frac{\sqrt{7}}{3}, \frac{\sqrt{7}}{3})$</td>
</tr>
<tr>
<td>$a_1 = a_2 = a_4, a_3$</td>
<td>N</td>
<td>N/Y</td>
<td>6</td>
<td>$(\frac{\sqrt{7}}{3}, \frac{\sqrt{7}}{3}, \frac{\sqrt{7}}{3}, \frac{\sqrt{7}}{3})$</td>
</tr>
<tr>
<td>As above</td>
<td>Y</td>
<td>Y</td>
<td>$\infty$</td>
<td>All such $(v, r, a_1, a_2, a_3, a_4)$</td>
</tr>
<tr>
<td>As above</td>
<td>Y</td>
<td>N</td>
<td>0</td>
<td>All such $(v, r, a_1, a_2, a_3, a_4)$</td>
</tr>
<tr>
<td>$a_1 = a_2, a_3 = a_4$</td>
<td>N/Y</td>
<td>N/Y</td>
<td>8</td>
<td>$(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})$</td>
</tr>
<tr>
<td>$a_1 = a_2, a_3, a_4$</td>
<td>N</td>
<td>N/Y</td>
<td>8</td>
<td>$(\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})$</td>
</tr>
<tr>
<td>As above</td>
<td>Y</td>
<td>N/Y</td>
<td>6</td>
<td>$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$</td>
</tr>
<tr>
<td>$a_1, a_2, a_3, a_4$</td>
<td>N/Y</td>
<td>N/Y</td>
<td>9</td>
<td>$(16, \frac{46}{10}, \frac{209}{10}, \frac{152}{1}, \frac{25}{1}, 32, 36)$</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
H^2 + J^2 + K^2 - A_1^2 + A_2^2 + A_3^2 + A_4^2 &= 0, \\
H^2(A_1^2A_3^2 + A_2^2A_4^2) + J^2(A_1^2A_2^2 + A_2^2A_3^2) + K^2(A_1^2A_2^2 + A_3^2A_4^2) - H^2J^2K^2 + 81V^4 &= 0, \\
2A_1A_2A_3A_4(\frac{1}{A_1^2} + \frac{1}{A_2^2} + \frac{1}{A_3^2} + \frac{1}{A_4^2}) &= 0, \\
[\sqrt{H, A_1, A_3}][H, A_2, A_4], \sqrt{J, A_1, A_4}][J, A_2, A_3], \sqrt{K, A_1, A_2}[K, A_3, A_4] &= 0.
\end{align*}
\]

\[
\frac{64(3V)^{10}R^2}{1} = 0.
\]

Let $f_1 = 0, f_2 = 0, f_3 = 0$ be the three equations above replacing with variables $r_1 = H^2, r_2 = J^2,$ $r_3 = K^2,$ and parameters $v = V^2, r = R^2, a_1 = A^2, a_2 = A^2, a_3 = A^2, a_4 = A^2$. The problem now is reduced to counting the number of positive common zeros $(r_1, r_2, r_3)$ of $f_1, f_2, f_3$ for all positive parameters $v, r, a_1, a_2, a_3, a_4$. Here are our main results.

**Theorem 1.** Given positive parameters $v, r, a_1, a_2, a_3, a_4$, we obtain various upper bounds $u$ on the numbers of complex common zeros for $(f_1, f_2, f_3)$ in Table 1, where $E_1, E_2$ denote the equations $27v^2 = a_2(a_3 - a_4)^2, 108vr = (3a_2 + a_3)^2$, respectively. In the example $\ast$, there are six common zeros with positive $r_1, r_2, and r_3$.

**Remark 1.** The cases of distinct $a_1 = a_2, a_3 = a_4$, distinct $a_1 = a_2, a_3, a_4$, and distinct $a_1, a_2, a_3, a_4$ have been proved in Yang and Zeng (2013) using different technique. For distinct $a_1 = a_2, a_3, a_4$ with $E_1$, we obtain a better bound, six. For distinct $a_1, a_2, a_3, a_4$, the upper bound, eight, obtained in Yang and Zeng (2013) is in the real domain.

**Remark 2.** In the cases of $a_1 = a_2 = a_3 = a_4$, as shown in McConnell (2012), there exists only one tetrahedron determined by volume, circumradius and face areas. This is also presented as an exercise in Mazur’s problem set in 1999 (Mazur, 1999). When $(v, r, a_1, a_2, a_3, a_4) = (\frac{\sqrt{7}}{3}, \frac{\sqrt{7}}{3}, \frac{\sqrt{7}}{3}, \frac{\sqrt{7}}{3}, \frac{\sqrt{7}}{3}, \frac{\sqrt{7}}{3})$, there is only one complex zero that is also real. It is the regular tetrahedron of unit edge lengths.

**Remark 3.** In the cases of $a_1 = a_2 = a_4 \neq a_3$ where $E_1$ and $E_2$ hold, we generalize the counter example by Yang and Zeng (2005). This is also obtained in McConnell (2012) by different method. There $v = 194481, r = \frac{1849}{127}, a_2 = 11907, a_3 = 21168$. As we fix the circumradius $r = \frac{1849}{127}$ without loss of generality, we note that $v$ is uniquely determined by $a_2, a_3$, but $(a_2, a_3)$ can be any zeros of the following polynomial

\[
f = -10256403a_3^2 + 81a_4^2 + 20512806a_2^2a_3 + 108a_2^3a_3 - 10256403a_2a_3^2 + 54a_2^2a_3^2 + 12a_2a_3^3 + a_4^3.
\]

The plot of the graph and a point $(a_2, a_3) = (11907, 21168)$ on it is given in Fig. 2.

**Remark 4.** The example given in each case where there are finitely many zeros can be numerically verified by applying the command N Solve in Mathematica. We will also rigorously prove it by Hermite’s root counting method. We present our implementation in Mathematica for computing the Hermite
matrices. Our implementation can be found in Tsai (2015) which is a collaborative work with Professor Rick Moeckel.

**Remark 5.** In the example *, we prove there are nine complex zeros. Moreover, using the signatures of various Hermite matrices, we prove, among the nine complex zeros, six of them are positive, which verifies the claim in Lisoněk and Israel (2000) that those parameters determine six tetrahedra. Similarly, we obtain four positive zeros in the example of \((v, r, a_1, a_2, a_3, a_4) = (\frac{1}{12}, \frac{7}{12}, \frac{7}{12}, \frac{7}{12}, \frac{7}{12}, \frac{7}{12})\). Taking symmetry into consideration, we obtain two essentially different tetrahedra shown in Fig. 1 as claimed in Lisoněk and Israel (2000).

3. Review of Groebner basis theory

### 3.1. Groebner basis

We study our polynomial systems mainly with Groebner basis tool. Such tool is invented by Professor Buchberger in 1965 (Buchberger, 1965). Now we review this tool. Some theorems about parametric systems and root counting theorems in the complex domain will be given in the next subsection.

Consider \(m\) polynomials in \(n\) variables \(f_1, \ldots, f_m \in \mathbb{F}[x_1, \ldots, x_n]\), where \(\mathbb{F}\) is a field. Consider the **ideal** generated by \(f_1, \ldots, f_m\) denoted by \(I = \langle f_1, \ldots, f_m \rangle\). Let \(f \in \mathbb{F}[x_1, \ldots, x_n]\). Then \(f \in I \iff \exists a_1, \ldots, a_m \in \mathbb{F}[x_1, \ldots, x_n]\) such that \(f = a_1 f_1 + \cdots + a_m f_m\). On the other hand, consider all the points \((x_1, \ldots, x_n)\) in \(\mathbb{F}^n\) that are common zeros of \(f_i\) for all \(i = 1, \ldots, m\). We call the set of all these points the **variety** defined by \(f_1, \ldots, f_m\) and denote it by \(V_\mathbb{F} = V_\mathbb{F}(f_1, \ldots, f_m)\). Our goal here is to count the points in \(V_\mathbb{F}\).

The ideal \(I\) gives an equivalence relation in \(\mathbb{F}[x_1, \ldots, x_n]\), i.e. \(f \sim g \iff f - g \in I\). Therefore, we can consider the quotient space determined by this equivalence relation and denote it by \(A = \mathbb{F}[x_1, \ldots, x_n]/I\). It is indeed an algebra over \(\mathbb{F}\) if we define the operations naturally. By studying the ideal \(I\) and quotient algebra \(A\), we gain information of \(V_\mathbb{F}\). Groebner bases serve as useful tools to study \(I\) and \(A\). We now start to give the definition of Groebner bases.

We start with some terminologies. A monomial in \(x_1, \ldots, x_n\) is an expression of the form \(x_1^{\alpha_1} \cdots x_n^{\alpha_n}\) with nonnegative integers \(\alpha_i\). Monomial order is a total ordering on \(\mathbb{Z}_{\geq 0}^n\) satisfying \(\alpha + \gamma > \beta + \gamma\) and \(\gamma \geq (0, \ldots, 0)\) if \(\alpha > \beta\) and \(\gamma \in \mathbb{Z}_{\geq 0}^n\). It gives monomials a total ordering which preserves the order under multiplication and has smallest monomial 1. Then, for any \(f \in \mathbb{F}[x_1, \ldots, x_n]\), it makes
Proposition 1. Let \( \langle f_1, \ldots, f_m \rangle \neq \emptyset \) in \( \mathbb{F}[x_1, \ldots, x_n] \) and \( V_\mathbb{F} \) in \( \mathbb{F}^n \). Fixing a monomial order, we have the following (Cox et al., 1992; Dummit and Foote, 2004).

1. There exists an algorithm to compute a Groebner basis \( G = \{ g_1, \ldots, g_l \} \) with respect to this monomial order. \( G \) also generates \( \langle f_1, \ldots, f_m \rangle \).
2. Let \( E \) be a field containing \( \mathbb{F} \), \( I \) be the ideal generated by \( f_1, \ldots, f_m \) in \( E[x_1, \ldots, x_n] \). Then \( G \) is a Groebner basis of \( I \) in \( E[x_1, \ldots, x_n] \) with the same order.

A monomial order on \( \mathbb{F}[x_1, \ldots, x_n] \) can be described in terms of a \( n \times n \) matrix \( M \) with entries in \( \mathbb{F} \). We write the rows of \( M \) as \( w_1, \ldots, w_n \). Given two monomials with exponent vectors \( \alpha \) and \( \beta \), we have \( \alpha > \beta \) if \( w_1 \cdot \alpha > w_1 \cdot \beta \). If \( w_1 \cdot \alpha = w_1 \cdot \beta \), we compare \( w_2 \cdot \alpha \) with \( w_2 \cdot \beta \). Repeat the procedure until we find which one is larger. If \( w = (1, \ldots, 1, 0, \ldots, 0) \), where the first \( (n-i+1) \) entries are 1 and the rest are 0, it is the graded reverse lexicographic order (grlex order).

Consider variables \( x_1, \ldots, x_n, a_1, \ldots, a_l \), we now define a special monomial order for those \( n+l \) variables. This monomial order is given by a \( (n+l) \times (n+l) \) matrix with two diagonal blocks and zeros outside these two blocks. The first \( n \times n \) block gives a monomial order matrix for \( x_i \)'s and the second \( l \times l \) block gives a monomial order matrix for \( a_j \)'s. Monomial orders defined by such matrices are called block orders. We denote by \( <_a \) the monomial order reduced from \( < \) on monomials in \( x_i \)'s. It is actually given by the matrix of the first \( n \times n \) block. The block orders are very useful when we deal with polynomial systems with parameters in the coefficients.

3.2. Parametric polynomial systems and upper bounds in the complex domain

Consider a parametric polynomial system with the same numbers of variables and polynomials and denote it by \( \mathcal{F} = \{ f_1, \ldots, f_n \} \), where \( f_i \in \mathbb{D}[x_1, \ldots, x_n] \) and \( \mathbb{D} = \mathbb{Q}[a_1, \ldots, a_l] \). We define the specialization at a point \( a \in \mathbb{C}^l \). It is a ring homomorphism \( \varphi^a : \mathbb{D}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n] \) such that \( \varphi^a(f) \) is the polynomial with complex coefficients after substituting the parameter with the value \( a \) for \( f \in \mathbb{D}[x_1, \ldots, x_n] \). Denote \( \varphi^a(f) = f^a \) and \( \varphiт(\mathcal{F}) = \{ f^a | f \in \mathcal{F} \} = \mathcal{F}^a \). We can view polynomials \( f_i \in \mathbb{Q}[x_1, \ldots, x_n, a_1, \ldots, a_l] \), form the ideal \( I = \langle f_1, \ldots, f_n \rangle \subseteq \mathbb{Q}[x_1, \ldots, x_n, a_1, \ldots, a_l] \), and obtain the property below.

Proposition 2. \( I^a = \langle f_1^a, \ldots, f_n^a \rangle \subseteq \mathbb{C}[x_1, \ldots, x_n] \), for all \( a \in \mathbb{C}^l \).

Proof. If \( f \in I^a \), \( f = \varphi^a(h_1 f_1 + \cdots + h_n f_n) \) for some \( h_i \in \mathbb{Q}[x_1, \ldots, x_n, a_1, \ldots, a_l] \). So, \( f = h_1^a f_1^a + \cdots + h_n^a f_n^a \). We have \( h_i^a \in \mathbb{C}[x_1, \ldots, x_n] \). Therefore, \( f \in \langle f_1^a, \ldots, f_n^a \rangle \). Conversely, if \( f = k_1 f_1^a + \cdots + k_n f_n^a \), where \( k_j \in \mathbb{C}[x_1, \ldots, x_n] \), then \( f = \varphi^a(k_1 f_1 + \cdots + k_n f_n) \) since \( k_j = k_j^a \) for all \( j, a \). □

Fix a block order \( < \) on \( x_1, \ldots, x_n, a_1, \ldots, a_l \), and compute a Groebner basis \( G \) in \( \mathbb{Q}[x_1, \ldots, x_n, a_1, \ldots, a_l] \). Then we have the following useful property (Gonzales-Vega et al., 2005). This provides us a way to compute generic Groebner bases for most of the parameters.

Proposition 3. Let \( h_i = \text{LC}_{<_a}(g_i) \in \mathbb{Q}[a_1, \ldots, a_l] \). \( G^a \) is a Groebner basis of \( I^a = \langle f_1^a, \ldots, f_n^a \rangle \) in \( \mathbb{C}[x_1, \ldots, x_n] \) for all \( a \in \mathbb{C}^l \setminus \mathbb{V}_\mathbb{C}(h) \), where \( h = h_1 \cdots h_l \).

Let \( a \in \mathbb{C}^l \), there may exist some \( g_i \in G \) such that \( h_i^a = (\text{LC}_{<_a}(g_i))^a = \text{LC}_{<_a}(g_i^a) = 0 \). In these cases, there is still a way using Hilbert functions to tell if \( G^a \) is a Groebner basis of \( I^a \) or not.
(Gonzales-Vega et al., 2005). However, if we do not need $G^a$ to be a Groebner basis of $I^a$, we do not have to go through the Hilbert functions check. Sometimes the following observation will be enough to serve our purpose.

**Proposition 4.** Given any $g_i \in G$ and $a \in \mathbb{C}^i$, we have $LT_{<c}(g_i^a) \in \langle LT(I^a) \rangle$ in $\mathbb{C}[x_1, \ldots, x_n]$.

Next, we list some properties regarding the number of the common complex zeros that we will apply in the next section. The proofs can be found in Cox et al. (1992).

**Proposition 5.** Let $I = \langle f_1, \cdots, f_n \rangle$ in $\mathbb{C}[x_1, \ldots, x_n]$, $A = \mathbb{C}[x_1, \ldots, x_n]/I$ and $V_C$ in $\mathbb{C}^n$. Fixing a monomial order, we have the following equivalent statements.

1. $V_C$ is a finite set.
2. $A$ is finite dimensional over $\mathbb{C}$.
3. For each $i$, $1 \leq i \leq n$, there is some $m_i \geq 0$ such that $x_i^{m_i} \in \langle LT(I) \rangle$.
4. Let $G$ be a Groebner basis for $I$. Then for each $i$, $1 \leq i \leq n$, there is some $m_i \geq 0$ such that $x_i^{m_i} = LM(g_i)$ for some $g_i \in G$.

Usually, we use the third or forth statements to verify whether a system has finitely many complex common zeros or not by computing Groebner bases. When finiteness is verified, we use the following properties to achieve an upper bound for the number of complex zero. Again, the proofs can be found in Cox et al. (1992).

**Proposition 6.** Let $I = \langle f_1, \cdots, f_n \rangle$ in $\mathbb{C}[x_1, \ldots, x_n]$, $A = \mathbb{C}[x_1, \ldots, x_n]/I$ and $V_C$ in $\mathbb{C}^n$ is a finite set. Fixing a monomial order, we have the following.

1. The number of points in $V_C$ is at most $\dim(A)$.
2. $A$ is isomorphic to $\text{Span}(\langle x^a : x^a \notin \langle LT(I) \rangle \rangle)$ as vector spaces over $\mathbb{C}$.

### 3.3. Hermite’s root counting method

Consider $I = \langle \mathcal{F} \rangle = \langle f_1, \cdots, f_n \rangle$ in $\mathbb{R}[x_1, \ldots, x_n]$, $A = \mathbb{R}[x_1, \ldots, x_n]/I$. Given $f \in A$, one can define the linear multiplication map on $A$ by $L(f)(g) := fg$. Let $g \in A$. The function defined by $A \rightarrow \mathbb{R} : f \mapsto \text{Trace}(L(gf^2))$ is a quadratic form on $A$. This is called the Hermite quadratic form determined by $\mathcal{F}$ and $g$. Given a basis of $A$, say $\{b_1, \ldots, b_m\}$, then we can form an $m \times m$ matrix representation of the quadratic form, called the Hermite matrix and denoted by $H(\mathcal{F}, g)$. Each entry $H(\mathcal{F}, g)(i, j)$ is given by $\text{Trace}(L(gb_ib_j))$.

From the basic facts about quadratic forms, we know the rank and the signature are independent of the choice of the basis. We have the following Hermite’s root counting theorem. The proof of this theorem can be found in Cohen et al. (1999).

**Proposition 7.** Matrix rank of $H(\mathcal{F}, g)$ equals to the number of complex roots of $\mathcal{F}$ with $g \neq 0$. Signature of $H(\mathcal{F}, g)$ equals to the number of real roots of $\mathcal{F}$ with $g > 0$ minus the number of real roots of $\mathcal{F}$ with $g < 0$.

Due to the advantages of Groebner bases, we can explicitly find the $m \times m$ real matrix representation of $L$ with respect to this basis $\{b_1, \ldots, b_m\}$. In particular, we can explicitly compute the trace. Therefore, we can explicitly compute the $m \times m$ real matrix $H(\mathcal{F}, g)$. Each entry is the trace of a $m \times m$ real matrix.

### 4. Proof

In this section, we present proofs of the Theorem 1 from section 2. Recall that in Theorem 1, we consider the parametric polynomial system $f_1 = f_2 = f_3 = 0$ in three variables $r_1$, $r_2$, $r_3$ and six parameters $(v, r, a_1, a_2, a_3, a_4)$. We divide the proofs for upper bounds into five cases from 4.1 to 4.5.
according to essentially different choices of distinct face areas. In 4.6, we consider examples that realize the upper bounds.

4.1. \(a_1 = a_2 = a_3 = a_4\)

Let \(a_1 = a_2 = a_3 = a_4\). Consider \(I = \langle f_1, f_2, f_3 \rangle \in \mathbb{Q}[r_1, r_2, r_3, v, r, a_2, a_3]\), and \(\mathcal{P} = \{(v, r, a_2) \in \mathbb{R}_+^2\}\). Fix the block order \(<\) such that the first block \(<r_1\) on \(r_1, r_2, r_3\) is a grlex order with \(r_1 > r_2 > r_3\) and the second block \(<v, r, a_2\) is a grlex order with \(v > r > a_2\). We compute and obtain a Groebner basis \(G = \{g_1, \ldots, g_4\}\). The four leading terms \(h_i\) are given below.

\[
\begin{align*}
    h_1 &= r_1, \\
    h_2 &= v^4r_2^2, \\
    h_3 &= v^4r_3^2, \\
    h_4 &= r_2^2r_3.
\end{align*}
\]

So, \(G^a\) is a Groebner basis of \(I^a\) for all \(a \in \mathcal{P}\). Moreover, \(\langle LT(I^a) \rangle = \{r_1, r_2^2, r_3^2, r_2^2r_3\}\). \(\{x^a : x^a \notin \langle LT(I^a) \rangle \} = \{1, r_3, r_2^2, r_2r_3, r_2^2r_3\}\). So, \(\dim(A^a) = C[r_1, r_2, r_3]/I^a) = 6\).

4.2. Distinct \(a_1 = a_2 = a_3, a_3\)

Let \(a_1 = a_2 = a_4 \neq a_3\). Consider \(I = \langle f_1, f_2, f_3 \rangle \in \mathbb{Q}[r_1, r_2, r_3, v, r, a_2, a_3]\), and \(\mathcal{P} = \{(v, r, a_2, a_3) \in \mathbb{R}_+^4 : a_2 \neq a_3\}\). Fix the block order \(<\) such that the first block \(<r_1\) on \(r_1, r_2, r_3\) is a grlex order with \(r_1 > r_2 > r_3\) and the second block \(<v, r, a_2, a_3\) is a grlex order with \(v > r > a_2 > a_3\). We compute and obtain a Groebner basis \(G = \{g_1, \ldots, g_4\}\). The four leading terms \(h_i\) are given below.

\[
\begin{align*}
    h_1 &= r_1, \\
    h_2 &= -3v^2(27v^2 - a_3(a_3 - a_2)^2)r_2^2, \\
    h_3 &= -3v^2(27v^2 - a_3(a_3 - a_2)^2)r_2^3, \\
    h_4 &= r_2^2r_3.
\end{align*}
\]

For \(27v^2 - a_3(a_3 - a_2)^2 \neq 0\), \(G^a\) is a Groebner basis of \(I^a\). Moreover, \(\langle LT(I^a) \rangle = \{r_1, r_2^2, r_3^2\}\) and \(\{x^a : x^a \notin \langle LT(I^a) \rangle \} = \{1, r_3, r_2^2, r_2r_3, r_2^2r_3\}\). Therefore, \(\dim(A^a) = C[r_1, r_2, r_3]/I^a) = 6\).

Next, consider \(27v^2 - a_3(a_3 - a_2)^2 = 0\). We compute the Groebner basis for \(\langle f_1, f_2, f_3, 27v^2 - a_3(a_3 - a_2)^2 \rangle \in \mathbb{Q}[r_1, r_2, r_3, v, r, a_2, a_3]\) using a grlex order with \(r_1 > r_2 > r_3 > v > r > a_2 > a_3\). We obtain \(g = -v^2(108r - (3a_2 + a_3)^2)\) in the basis. If \(108r - (3a_2 + a_3)^2 \neq 0\), we conclude that the system \(\{f_1, f_2, f_3, 27v^2 - a_3(a_3 - a_2)^2 = 0\), and \(108r - (3a_2 + a_3)^2 \neq 0\).

Finally, if \(27v^2 - a_3(a_3 - a_2)^2 = 0\) and \(108r - (3a_2 + a_3)^2 = 0\), then \(v^2 = \frac{1}{27}a_2(a_2 - a_3)^2\) and \(rv = \frac{3(a_2 + a_3)^2}{108}\). Note \(f_8\) has \(v^2\) in the coefficient and \(f_9\) has \(rv^5\) in the coefficient. We replace \(v^2\) and \(rv^5\) by \(\frac{1}{27}a_2(a_2 - a_3)^2\) and \(\frac{3(a_2 + a_3)^2}{108}\), respectively. We get a system \(I = \langle f_1, f_2, f_3 \rangle \in \mathbb{Q}[r_1, r_2, r_3, a_2, a_3]\). Fix the block order \(<\) such that the first block \(<r_1\) on \(r_1, r_2, r_3\) is a grlex order with \(r_1 > r_2 > r_3\) and the second block \(<v, r, a_2\) is a grlex order with \(a_2 > a_3\). We compute and obtain a Groebner basis \(G = \{r_1 + r_2 + a_2 - a_3, -r_2^2r_3 - r_2^2r_3 + 3r_2r_3a_2 - 4a_2^2 + r_2r_3a_3 + 8a_2a_3 - 4a_2^2a_3\}\). Leading terms are \(r_1\) and \(-r_2^2r_3\). Therefore, for all \(a_2, a_3, G^a_{a_2, a_3}\) is a Groebner basis for \(I^a_{a_2, a_3}\). By Proposition 5, the dimension of \(A^a_{a_2, a_3}\) is infinite and \(V_{C}(I^a_{a_2, a_3})\) contains infinitely many points.

4.3. Distinct \(a_1 = a_2, a_3 = a_4\)

Note that the claim in this case has been proved by Yang and Zeng (2013) using different method. Here, we provide another proof using a uniform technique that we use for all the cases.

Let \(a_1 = a_2 \neq a_3 = a_4\). Consider \(I = \langle f_1, f_2, f_3 \rangle \in \mathbb{Q}[r_1, r_2, r_3, v, r, a_2, a_3]\), and \(\mathcal{P} = \{(v, r, a_2, a_3) \in \mathbb{R}_+^4 : a_2 \neq a_3\}\). Fix the block order \(<\) such that the first block \(<r_1\) on \(r_1, r_2, r_3\) is a grlex order with
and the second block $<2$ on $v, r, a_2, a_3$ is a grlex order with $v > 2 r > 2 a_2 > 2 a_3$. We compute and obtain a Groebner basis $G = \{g_1, \ldots, g_6\}$. The six leading terms $h_i$ are given below.

$$
\begin{align*}
  h_1 &= v^2 r_1 f_1, \\
  h_2 &= r^2 f_2, \\
  h_3 &= v^2 (a_2 - a_3) (a_2 - a_4) r_2 f_3, \\
  h_4 &= v^4 (a_2 - a_3) (a_2 - a_4) r_2 f_4, \\
  h_5 &= v^6 (a_2 - a_3) (a_2 - a_4) r_2 f_5, \\
  h_6 &= v^8 r_2 f_6.
\end{align*}
$$

So, $G^a$ is a Groebner basis of $I^a$ for all $a \in \mathcal{P}$. And, $\langle \{I^a\} \rangle = \{r_1 f_1, r_2 f_2, r_3 f_3, r_4 f_4, r_2 r_3 f_5, r_2 r_3 f_6\}$ and $\{x^a : x^a \notin \langle I^a \rangle \} = \{1, r_3 f_1, r_3 f_2, r_2 r_3 f_1, r_2 r_3 f_2\}$. Therefore, we obtain that $\dim(A^a = C[r_1, r_2, r_3] / I^a) = 8$.

### 4.4. Distinct $a_1 = a_2, a_3, a_4$

Note that the upper bound eight has been proved by Yang and Zeng (2013) using different method. Here, we provide another proof using a uniform technique that we use for all the cases. Especially, when $E_1$ is satisfied, we obtain a better bound, six.

Let $\{a_1 = a_2, a_3, a_4\}$ be distinct. Consider $I = (f_1, f_2, f_3) \in \mathbb{Q}[r_1, r_2, r_3, v, r, a_2, a_3, a_4]$, $\mathcal{P} = \{(v, r, a_2, a_3, a_4) \in \mathbb{R}_+^5 : a_2, a_3, a_4 \text{ are pairwise distinct}\}$.

Fix the block order $< \text{ such that the first block } <1 \text{ on } r_1, r_2, r_3 \text{ is a grlex order with } r_1 > 1 r_2 > 1 r_3 \text{ and the second block } <2 \text{ on } v, r, a_2, a_3, a_4 \text{ is a grlex order with } v > 2 r > 2 a_2 > 2 a_3 > 2 a_4. \text{ We compute and obtain a Groebner basis } G = \{g_1, \ldots, g_7\}.$ Consider five leading terms $h_i = IT_{<1} (g_i)$ below.

$$
\begin{align*}
  h_1 &= v^2 r_1 f_1, \\
  h_2 &= r^2 f_2, \\
  h_3 &= v^2 (a_2 - a_3) (a_2 - a_4) r_2 f_3, \\
  h_4 &= v^4 (a_2 - a_3) (a_2 - a_4) (27v^2 - a_2 (a_3 - a_4))^2 f_4, \\
  h_5 &= v^6 (a_2 - a_3) (a_2 - a_4) r_2 f_5, \\
  h_6 &= v^8 r_2 f_6.
\end{align*}
$$

If $27v^2 - a_2 (a_3 - a_4)^2 \neq 0$, we obtain monomials $r_1, r_2^2 f_3, r_2^2, r_2 f_4, r_2 f_5 r_3^2$ in $\langle I^a \rangle$. Therefore, $\{x^a \notin \langle I^a \rangle \} \text{ is contained in } \{1, r_3 f_1, r_3 f_2, r_2 r_3 f_1, r_2 r_3 f_2\}.$ And, $V_{C^2} (I^a)$ contains at most eight points.

Next, consider $27v^2 - a_2 (a_3 - a_4)^2 = 0$. This is equivalent to $a_2 = \frac{27v^2}{(a_3 - a_4)^2}$. We replace $a_2$ with $\frac{27v^2}{(a_3 - a_4)^2}$ in $f_1, f_2, f_3$ and consider $I = (f_1, f_2, f_3) \in \mathbb{Q}[r_1, r_2, r_3, v, r, a_3, a_4]$ and $\mathcal{P} = \{(v, r, a_3, a_4) \in \mathbb{R}_+^3 : a_3 \neq a_4, \frac{27v^2}{(a_3 - a_4)^2} \neq a_3, \frac{27v^2}{(a_3 - a_4)^2} \neq a_4\}$. Fix the block order $< \text{ such that the first block } <1 \text{ on } r_1, r_2, r_3 \text{ is a grlex order with } r_1 > 1 r_2 > 1 r_3 \text{ and the second block } <2 \text{ on } v, r, a_3, a_4 \text{ is a grlex order with } v > 2 r > 2 a_3 > 2 a_4. \text{ We compute and obtain a Groebner basis } G = \{g_1, \ldots, g_42\}.$ Consider the following seven leading terms $h_i$.

$$
\begin{align*}
  h_1 &= (a_3 - a_4)^2 r_1, \\
  h_2 &= 4v^4 (a_3 - a_4)^6 (27v^2 - a_3^2 + 2a_3^2 a_4 - a_3 a_4^2) (27v^2 - a_3 a_4 + 2a_3 a_4^2 - a_3^3) r_2, \\
  h_7 &= -v^4 (a_3 - a_4)^6 c_2 r_2^2 f_3, \\
  h_11 &= (a_3 - a_4)^4 r_2 f_3, \\
  h_17 &= 5832 v^{16} r_1 f_2, \\
  h_18 &= 2916 v^{14} r_1^2 f_2, \\
  h_20 &= -4v^4 (a_3 - a_4)^6 (27v^2 - a_3^2 + 2a_3^2 a_4 - a_3 a_4^2) (27v^2 - a_3^2 a_4 + 2a_3 a_4^2 - a_3^3) c_1 r_3.
\end{align*}
$$

In this list, since we consider $a \in \mathcal{P}$, we know $27v^2 - a_3^3 + 2a_3^2 a_4 - a_3 a_4^2 \neq 0$, $27v^2 - a_3^2 a_4 + 2a_3 a_4^2 - a_3^3 \neq 0$. It turns out that the following factors $c_1$ and $c_2$ appear in $h_{20}$ and $h_7$, respectively:
\[ c_1 = 216v^2a_3 + a_3^4 + 216v^2a_4 - 4a_3^2a_4 + 6a_2^2a_2^2 - 4a_3a_4^3 + a_4^4 - 108rv(a_3 - a_4)^2 \]
\[ c_2 = (54v^2 + a_3^2 - a_2^2a_4 - a_3a_4^2 + a_4^2)^2 - 108rv(a_3 - a_4)^4. \]

We will claim that \( c_1 \neq 0 \) if \( f_1^a, f_2^a, f_3^a \) have common zeros for \( a \in \mathcal{P} \). And, we will prove, whether \( c_2 \) is equal to zero or not, \( V_C(I^a) \) always contains at most six points.

Let’s prove \( c_1 \neq 0 \) first. Computing a Groebner basis \( G \) for the system \( \{ I, c_1 \} \in \mathbb{Q}[r_1, r_2, r_3, v, r, a_3, a_4] \) using the grlex order with \( r_1 > r_2 > r_3 > v > r > a_3 > a_4 \), we obtain a polynomial \( v^{20}(r_1 + r_2 + r_3 - 3a_3 - a_4) \) in the basis (the 242nd element in \( G \) with 441 polynomials). Note that \( f_1 = r_1 + r_2 + r_3 - 2a_2 - a_3 - a_4 \). This means either \( a_3 + 3a_4 = 2a_2 + a_3 + a_4 \) or \( 3a_3 + a_4 = 2a_2 + a_3 + a_4 \), that is, either \( a_4 = a_2 \) or \( a_3 = a_2 \), which is impossible. Therefore, \( c_1 \neq 0 \) if \( I^a \) has zeros for \( a \in \mathcal{P} \).

In the case that \( c_2 \neq 0 \), we are ready to conclude the finiteness of \( V_C(I^a) \). From the list of 7 \( h_i \)'s, we obtain monomials \( r_1, r_2^2, r_2^2r_3, r_1r_2^2, r_1^2r_2, r_1^3r_2^3 \) in \( \langle LT(I^a) \rangle \). Therefore, \( \{ x^{a^2} : x \notin \langle LT(I^a) \rangle \} \) is contained in \( \{ 1, r_1, r_2^2, r_2, r_2^3 \} \). And, \( V_C(I^a) \) contains at most six points.

Finally, let’s consider \( c_2 = 0 \). This is equivalent to \( r = \frac{(54v^2 + a_3^2 - a_2^2a_4 - a_3a_4^2 + a_4^2)^2}{108v(a_3 - a_4)^5} \). Replacing \( r \) with the right hand side of the equation in our system, we obtain \( I \in \mathbb{Q}[r_1, r_2, r_3, v, a_3, a_4] \). Now, \( \mathcal{P} = \{ (v, a_3, a_4) \in \mathbb{R}^3_+ : a_3 \neq a_4, \frac{27v^2}{(a_3 - a_4)^2} \neq a_3 - \frac{27v^2}{(a_3 - a_4)^2} \neq a_4 \} \).

Fix the block order \( < \) such that the first block \( < 1 \) on \( r_1, r_2, r_3 \) is a grlex order with \( r_1 > r_2 > r_3 \) and the second block \( < 2 \) on \( v, a_3, a_4 \) is a grlex order with \( v > a_3 > a_4 \). We compute and obtain a Groebner basis \( G = \{ g_1, \ldots, g_{25} \} \). In this set, every leading term contains non-vanishing coefficient for all \( a \in \mathcal{P} \). Therefore, \( G^a \) is a Groebner basis of \( I^a \) for all \( a \in \mathcal{P} \). We look at the following to find the upper bound.

\[
\begin{align*}
\{ h_1 & = (a_3 - a_4)^2r_1, \\
\{ h_2 & = v^4(a_3 - a_4)^6(27v^2 - a_2^2a_4 - a_3a_4^2)(27v^2 - a_2^2a_4 + 2a_3a_4^2 - a_4^2)r_2, \\
\{ h_7 & = v^4(a_3 - a_4)^8(27v^2 - a_2^2a_4 - a_3a_4^2)(27v^2 - a_2^2a_4 + 2a_3a_4^2 - a_4^2)r_3. \\
\end{align*}
\]

So, we get \( \langle LT(I^a) \rangle = \{ r_1, r_2^2, r_3^2 \} \) and \( \{ x^{a^2} : x \notin \langle LT(I^a) \rangle \} = \{ 1, r_1, r_2^2, r_2, r_2^3, r_1^2r_2, r_1^3r_2^3 \} \). Therefore, \( \text{dim}(A^a) = \mathbb{C}[r_1, r_2, r_3]/I^a \) = 6.

4.5. Distinct \( a_1, a_2, a_3, a_4 \)

Note that the upper bound eight has been proved by Yang and Zeng (2013) using different method. Here, we provide another proof using a uniform technique that we use for all the cases. The upper bound nine obtained here is in the complex domain.

Let \( I = \langle f_1, f_2, f_3 \rangle \) in the polynomial ring \( \mathbb{Q}[r_1, r_2, r_3, v, r, a_1, a_2, a_3, a_4] \), and we have \( \mathcal{P} = \{ (v, r, a_1, a_2, a_3, a_4) \in \mathbb{R}^6_+ : a_1, a_2, a_3, a_4 \text{ are pairwise distinct} \} \). We will show that \( \text{dim}(A^a) = \mathbb{C}[r_1, r_2, r_3]/I^a \) = 9, and, therefore, \( V_C(I^a) \) contains at most nine points, for all \( a \in \mathcal{P} \).

Fix the block order \( < \) such that the first block \( < 1 \) on \( r_1, r_2, r_3 \) is a grlex order with \( r_1 > r_2 > r_3 \) and the second block \( < 2 \) on \( v, a_1, a_2, a_3, a_4 \) is a grlex order with \( v > a_1 > a_2 > \ldots > a_4 \). We compute and obtain a Groebner basis \( G = \{ g_1, \ldots, g_7 \} \). The seven leading terms \( h_i = \langle LT \rangle_{<1}(g_i) \) are given below.

\[
\begin{align*}
\{ h_1 & = r_1, \\
\{ h_2 & = r_2^2r_3, \\
\{ h_3 & = v^2(a_2 - a_3)(a_1 - a_4)r_3^2, \\
\{ h_4 & = v^2(a_2 - a_3)(a_1 - a_4)r_2r_3^2, \\
\{ h_5 & = v^2(a_1 - a_2)(a_1 - a_3)(a_2 - a_4)(a_3 - a_4)r_5^3, \\
\{ h_6 & = v^2(a_1 - a_2)(a_3 - a_4)r_2r_3^4, \\
\{ h_7 & = v^2(a_2 - a_3)(a_2 - a_4)(a_3 - a_4)r_2r_3^4. \\
\end{align*}
\]
Therefore, by Proposition 3, \(G^a\) is a Groebner basis of \(I^a\) for all \(a \in \mathcal{P}\). Also, \([LT(P)] = \{r_1, r_2^2r_3, r_2^3, r_2^2, r_3^2\}\), \(\{x^\alpha : x^\alpha \notin [LT(P)]\} = \{1, r_3^3, r_3^4, r_2, r_2r_3, r_2^2, r_2^3, r_2^4\}\). Therefore, \(\dim(A^a = \mathbb{C}[r_1, r_2, r_3]/I^P) = 9\).

4.6. Examples

Here, we only show the case when \((v, r, a_1, a_2, a_3, a_4) = (16, \frac{461}{16}, \frac{261}{4}, \frac{125}{4}, 32, 36)\). For all of the other examples of finite non-zero upper bounds, the numbers of complex zeros and positive zeros can be obtained in the same way.

Now \(I = (\mathcal{F}) = (f_1, f_2, f_3) \subseteq \mathbb{Q}[r_1, r_2, r_3]\) and \(A = \mathbb{R}[r_1, r_2, r_3]/I\). Computing the Groebner basis of \(I\) with respect to the graded reverse lexicographic order where \(r_1 > r_2 > r_3\), we get leading monomials \(2r_1, 32r_2^2r_3, 11232r_3^2, 11232r_2r_3^2\), \(43990016r_3^3\). There are nine monomials that are not divided by them. They are \(\{1, r_3^3, r_3^4, r_2, r_2r_3, r_2r_3^2, r_2^2, r_2^3, r_2^4\}\). So, the dimension of \(A\) is nine.

Next, we compute eight Hermite matrices \(H(\mathcal{F}, 1), H(\mathcal{F}, r_1), H(\mathcal{F}, r_2), H(\mathcal{F}, r_3), H(\mathcal{F}, r_1r_2), H(\mathcal{F}, r_1r_3), H(\mathcal{F}, r_2r_3), H(\mathcal{F}, r_1r_2r_3)\) and \(H(\mathcal{F}, r_1^2r_3)\). Computing the determinant of \(H(\mathcal{F}, 1)\), we found it is non-zero. Therefore, it has rank 9. Therefore, by Proposition 7, the number of complex zero is nine.

For the number of positive zeros, we compute eight signatures \(9, 9, 7, 7, 5, 5, 3, 3\) from the eight Hermite matrices. \(s_1, \cdots, s_8\) denote the number of real roots \((r_1, r_2, r_3)\) having the signs \((+, +, +), \cdots, (−, −, −)\), respectively. We get

\[
9 = s_1 + s_2 + s_3 + s_4 + s_5 + s_6 + s_7 + s_8,
9 = s_1 + s_2 + s_3 + s_4 - s_5 - s_6 - s_7 - s_8,
7 = s_1 + s_2 - s_3 - s_4 + s_5 + s_6 - s_7 - s_8,
5 = s_1 - s_2 + s_3 - s_4 + s_5 - s_6 + s_7 - s_8,
7 = s_1 + s_2 - s_3 - s_4 - s_5 - s_6 + s_7 + s_8,
5 = s_1 - s_2 + s_3 - s_4 - s_5 + s_6 - s_7 + s_8,
3 = s_1 - s_2 - s_3 + s_4 + s_5 - s_6 - s_7 + s_8,
3 = s_1 - s_2 - s_3 + s_4 + s_5 - s_6 - s_7 + s_8.
\]

Solving this linear system, we get \((s_1, \cdots, s_8) = (6, 2, 1, 0, 0, 0, 0, 0)\). In particular, \(s_1 = 6\). So, there are six positive roots of \(f_1 = f_2 = f_3 = 0\) when \((v, r, a_1, a_2, a_3, a_4) = (16, \frac{461}{16}, \frac{261}{4}, \frac{125}{4}, 32, 36)\). Numerically solving the system, we also obtain six positive zeros, \((r_1, r_2, r_3) \approx (10, 77, 78), (11, 105, 49), (116, 37, 11), (115, 12, 38), (46, 109, 10), and (74, 10, 80)\).

5. Conclusion and future studies

In this paper, we study the numbers of tetrahedra determined by different sets of volume, circumradius, and four face areas. We obtain parametric polynomial systems and reduce the problem to a root counting problem. Our system has three equations, three variables, and six parameters. Six parameters represent volume, circumradius, and four face areas. The goal is to estimate the number of tetrahedra determined for all positive parameters.

In our main results, Theorem 1, we identify parameters that determine infinitely many complex zeros and found various upper bounds, zero, six, eight, and nine for parameters that determine finitely many complex zeros. All of our bounds are sharp in the complex domain.

In order to give realization of a determined tetrahedron, we need to find solutions in the systems that are real and positive numbers in all coordinates. In Yang and Zeng (2013), the upper bound six is conjectured. In our numerical experiments, we did not find parameters with more than six positive zeros either. On the other hand, in Lisoněk and Israel (2000), Lisoněk and Israel find, when \(V = 16, R = \frac{461}{16}, A_1 = \frac{261}{4}, A_2 = \frac{125}{4}, A_3 = 32, A_4 = 36\), they determine six tetrahedra, which is also verified in section 4.6. Proving the upper bound six can be a good direction for future study.
In this paper, we did not use comprehensive Gröbner basis and relative quantifier free formula over reals. The first reason is considering the complexity of the system. Some attempts of the computations of comprehensive Gröbner basis and relative quantifier free formula did not return any result after a reasonable period of time. The second reason is since all of our Gröbner basis computations took less than two minutes, we believe our method provides some interesting applications in using Gröbner basis. Finally, considering the open questions posed by Lisoněk and Israel in 2000, a full description of various upper bounds will answer their questions. However, a more refined classifications where the exact numbers of complex or even positive zeros are given for all positive parameters would be much better. These will also be good directions for future studies.

References